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The No-Triangle Hypothesis for $\mathcal{N} = 8$ Supergravity

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ABSTRACT: We study the perturbative expansion of $\mathcal{N} = 8$ supergravity in four dimensions from the viewpoint of the “no-triangle” hypothesis, which states that one-loop graviton amplitudes in $\mathcal{N} = 8$ supergravity only contain scalar box integral functions. Our computations constitute a direct proof at six-points and support the no-triangle conjecture for seven-point amplitudes and beyond.

KEYWORDS: Models of Quantum Gravity, Supergravity Models.

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1. Introduction

$\mathcal{N} = 8$ supergravity [1] is a remarkable theory, being the maximally supersymmetric field theory containing gravity that is consistent with unitarity. It is a beautiful but complicated theory containing (massless) particles of all spins (≤ 2) whose interactions are constrained by a large symmetry group.

This article explores the perturbative expansion of this theory. It has been postulated that the perturbative expansion of this theory is more akin to that of $\mathcal{N} = 4$ super Yang-Mills theory than expected from its known symmetries. In particular, it is hypothesised that the one-loop amplitudes can be expressed as scalar box functions with rational coefficients [2]. We provide considerable evidence for this “no-triangle hypothesis” by examining the behaviour of physical on-shell amplitudes.

This dramatic simplification of the one-loop amplitudes is presumably a signature of an undiscovered symmetry or principle present in $\mathcal{N} = 8$ supergravity. These simplifications do not occur on a “diagram by diagram” basis in any current expansion scheme, instead they arise only when the diagrams are summed. Theories of supergravity in four dimensions are one (and two) loop finite [3]. Since the box functions are UV finite, the simplifications we see are certainly consistent with these arguments. However the cancellations are considerably stronger than they demand: for example theories with $\mathcal{N} < 8$ supergravity are UV finite at one-loop but the one-loop amplitudes are not merely box functions.

In this article, we consider one-loop amplitudes and care must be used in extending the implications beyond one-loop. However, we do expect the higher loops to have a softer UV structure than previously thought [4]. This opens the door to the possibility that $\mathcal{N} = 8$ supergravity may, like $\mathcal{N} = 4$ super Yang-Mills, be a finite theory in four dimensions.

2. The No-triangle Hypothesis

2.1 Background: One-loop Amplitudes

First we review the general structure of one-loop amplitudes in theories of massless particles. Consider the general form for an n -point amplitude obtained from, for example, a Feynman diagram calculation ¹,

$$M_n^{1\text{-loop}}(1, \dots, n) = \sum_{\text{Feynman diagrams}} I_r[P^m(l, \{k_i, \epsilon_i\})], \quad (2.1)$$

where each I_r is a loop momentum integral with r propagators in the loop and numerator $P^m(l, \{k_i, \epsilon_i\})$. Here k_i denotes the external (massless) momenta, ϵ_i denotes

¹For simplicity we restrict ourselves to covariant gauges with Feynman gauge-like propagators $\sim 1/p^2$

the polarisation tensors of the external states and l denotes the loop momentum. For clarity we suppress the k_i and ϵ_i labels. In general the numerator is a polynomial of degree m in the loop momentum. The value of m depends on the theory under consideration. The summation is over all possible diagrams.

We choose to organise the diagrams according to the number of propagators in the loop, r . For $r = n$ the integral will have only massless legs, while for $r < n$ at least one of the legs attached to the loop will have momentum, $K = k_a + \dots k_b$, which is not null, $K^2 \neq 0$. We will call these massive legs (although it is a slight misnomer in a purely massless theory).

An important technique for dealing with these integrals is that of Passarino-Veltman Reduction [5] which reduces any r -point integral to a sum of $(r - 1)$ point integrals ($r > 4$),

$$I_r[P^m(l)] \longrightarrow \sum_i I_{r-1}^i[P^{m-1}(l)]. \quad (2.2)$$

We will be evaluating the loop momentum integrals by dimensional regularisation in $D = 4 - 2\epsilon$ and working to $\mathcal{O}(\epsilon)$. In the reduction the degree of the loop momentum polynomial is also reduced by 1 from m to $(m - 1)$. The $(r - 1)$ point functions appearing are those which may be obtained from I_r by contracting one of the loop legs. This process can be iterated until we obtain four point integrals,

$$I_r[P^m(l)] \longrightarrow \sum_i I_4^i[P^{m-(r-4)}(l)]. \quad (2.3)$$

The four point integrals reduce

$$I_4^i[P^{m'}(l)] \longrightarrow c_i I_4^i[1] + \sum_j I_3^j[P^{m'-1}(l)], \quad (2.4)$$

where we now have the “scalar box functions”, $I_4[1]$, whose loop momentum polynomials are just unity. The coefficients c_i are rational functions of the momentum invariants of the amplitude (By rational we really mean non-logarithmic, since these coefficients may contain Gram determinants.) Similarly, reduction of polynomial triangles gives scalar triangles plus tensor bubble integral functions,

$$I_3^j[P^m(l)] \longrightarrow d_j I_3^j[1] + \sum_k I_2^k[P^{m-1}(l)]. \quad (2.5)$$

Finally we can express the tensor bubbles as scalar bubble functions plus rational terms,

$$I_2^k[P^m(l)] = e_k I_2^k[1] + R + \mathcal{O}(\epsilon) \quad (2.6)$$

Consequently any one-loop amplitude can be reduced to the form,

$$M_n^{1\text{-loop}}(1, \dots, n) = \sum_{i \in \mathcal{C}} c_i I_4^i + \sum_{j \in \mathcal{D}} d_j I_3^j + \sum_{k \in \mathcal{E}} e_k I_2^k + R + \mathcal{O}(\epsilon), \quad (2.7)$$

where the amplitude has been split into a sum of integral functions with rational coefficients and a rational part. The sums run over bases of box, triangle and bubble integral functions: \mathcal{C} , \mathcal{D} and \mathcal{E} . Which integral functions appear in a specific case will depend on the theory and process under consideration, as will be discussed below.

2.2 $\mathcal{N} = 4$ Super Yang-Mills Amplitudes

For Yang-Mills amplitudes the three-point vertex is linear in momentum, so generically an r -point integral function has a loop momentum polynomial of degree r . In general, a Passarino-Veltman reduction gives one-loop amplitudes containing all possible integral functions.

For supersymmetric theories cancellations between the different types of particle circulating in the loop lead to a reduction in the order of the loop momentum polynomial. For $\mathcal{N} = 4$ super Yang-Mills amplitudes, formalisms exist where four powers of loop momentum cancel and the generic starting point for the reduction is a polynomial of degree $m = (r - 4)$. This implies that the amplitude consists only of box and higher point integrals which, via a Passarino-Veltman reduction (2.3), give a very restricted set of functions: namely scalar box-functions,

$$A_{\mathcal{N}=4}^{1\text{-loop}} = \sum_{i \in \mathcal{C}} c_i I_4^i. \quad (2.8)$$

These cancellations can be more or less transparent depending on the gauge fixing and computational scheme employed. In general the manifest diagram by diagram cancellation is less than the maximal four powers. Schemes in which these cancellations are manifest include the Bern-Kosower string based rules [6] (where technically the cancellation occurs at the level of Feynman parameter polynomials) and well chosen background field gauge schemes [7]. In less favourable schemes cancellations between diagrams occur relatively late in the calculation.

2.3 $\mathcal{N} = 8$ Supergravity Amplitudes

Computation schemes for gravity calculations tend to be rather more complicated than for Yang-Mills as the three-point vertex is quadratic in momenta and so the loop momentum polynomial is of degree $2r$ [8]. For maximal supergravity we expect to see considerable cancellations.

In string theory, closed strings contain gravity and open strings contain gauge theories, so the heuristic relation,

$$\text{closed string} \sim (\text{open-string}) \times (\text{open-string}), \quad (2.9)$$

suggests a relationship between amplitudes of the form,

$$\text{gravity} \sim (\text{Yang-Mills}) \times (\text{Yang-Mills}), \quad (2.10)$$

in the low energy limit. For tree amplitudes this relationship is exhibited by the Kawai-Lewellen-Tye relations [9]. Even in low energy effective field theories for gravity [10] the KLT-relations can be seen to link effective operators [11]. The KLT-relations also hold regardless of massless matter content [12]. For one-loop amplitudes we expect such relations for the *integrands* of one-loop amplitudes rather than the amplitudes themselves. Indeed, the equivalent of the Bern-Kosower rules for gravity [13, 14] give an initial loop momentum polynomial of degree,

$$2r - 8 = 2(r - 4). \quad (2.11)$$

This power counting is consistent with the heuristic expectation of string theory.

Using this power counting, reduction for $r > 4$ leads to a sum of tensor box integrals with integrands of degree $r - 4$ which would then reduce to scalar boxes *and* triangle, bubble and rational functions,

$$M_{\mathcal{N}=8}^{1\text{-loop}} = \sum_{i \in \mathcal{C}} c_i I_4^i + \sum_{j \in \mathcal{D}} d_j I_3^j + \sum_{k \in \mathcal{E}} e_k I_2^k + R, \quad (2.12)$$

where we expect that the triangle functions I_3 are present for $n \geq 5$, the bubble functions I_2 for $n \geq 6$ and the rational terms for $n \geq 7$. Note that functions, other than the scalar boxes, only appear after reduction.

2.4 The No-triangle Hypothesis

The “No-triangle hypothesis” states that any one-loop amplitude of $\mathcal{N} = 8$ supergravity is a sum of box integral functions multiplied by rational coefficients,

$$M_{\mathcal{N}=8}^{1\text{-loop}} = \sum_{i \in \mathcal{C}} c_i I_4^i. \quad (2.13)$$

The hypothesis originates from explicit computations which show that despite the previous power counting arguments, one-loop amplitudes for $\mathcal{N} = 8$ supergravity have a form analogous to that of one-loop $\mathcal{N} = 4$ super Yang-Mills amplitudes.

The first definite calculation of a one-loop amplitude for both $\mathcal{N} = 4$ super Yang-Mills and $\mathcal{N} = 8$ supergravity was performed by Green, Schwarz and Brink [15]. By taking the low energy limit of string theory, they obtained the four point one-loop amplitudes:

$$\begin{aligned} A_{(1,2,3,4)}^{1\text{-loop}} &= st \times A_{(1,2,3,4)}^{\text{tree}} \times I_4(s, t), \\ M_{(1,2,3,4)}^{1\text{-loop}} &= stu \times M_{(1,2,3,4)}^{\text{tree}} \left(I_4(s, t) + I_4(s, u) + I_4(t, u) \right). \end{aligned} \quad (2.14)$$

Here $I_4(s, t)$ denotes the scalar box integral with attached legs in the order 1234 and s , t and u are the usual Mandelstam variables. The above Yang-Mills amplitude is the leading in colour contribution. For gravity amplitudes we suppress factors of κ (

κ^{n-2} for tree amplitudes and κ^n for one-loop amplitudes.) Although only composed of boxes, this gravity amplitude is consistent with the power counting of $2(r-4)$ with $r \leq n$.

Beyond four-point we expect to find contributions from other integral functions in addition to the boxes. However in ref. [16] the five and six-point MHV^2 amplitudes were evaluated using unitarity techniques and shown to consist solely of box integral functions. It was conjectured that this behaviour continued to all MHV amplitudes and an all- n ansatz consisting of box functions was presented. This ansatz was also consistent with factorisation. In ref. [2] it was postulated that this was a general feature of $\mathcal{N} = 8$ amplitudes. In ref. [17] the hypothesis was explored for the six-point $NMHV$ amplitude and it was shown that the boxes alone gave the correct IR behaviour of the amplitude.

In this paper we aim to present further evidence in favour of the “no-triangle hypothesis”. While we fall short of presenting a proof, we feel that the weight of evidence is compelling. The evidence is based on IR structure, unitarity and factorisation. In the six-point case this evidence does constitute a proof.

3. Evidence For The No-triangle Hypothesis

We use a range of techniques to study different parts of the amplitude: unitarity, factorisation and the singularity structure of the on-shell physical amplitudes. Our arguments are complete for $n \leq 6$ point amplitudes. Fortunately there has been considerable progress in computing one-loop amplitudes inspired by the duality with twistor space [18–31]: we will freely use many of these new techniques.

We use arguments based on the IR divergences of the amplitude to conclude that the one and two-mass triangles must vanish. We use a study of the two-particle cuts to deduce that the bubble integrals are absent and, by numerically examining triple cuts, we show that the coefficients of three-mass triangles vanish. Finally we use factorisation arguments to discuss the rational pieces of the amplitude.

3.1 IR: Soft Divergences

The expected soft divergence of an n -point one-loop graviton amplitude [32] is,

$$M_{(1,2,\dots,n)}^{\text{one-loop}} \Big|_{\text{soft}} = \frac{i\kappa^2}{(4\pi)^2} \left[\frac{\sum_{i<j} s_{ij} \ln[-s_{ij}]}{2\epsilon} \right] \times M_{(1,2,\dots,n)}^{\text{tree}}. \quad (3.1)$$

²Amplitudes are conveniently organised according to the number of negative helicity external states. For amplitudes with “all-positive” or “one negative the remaining positive” helicity configurations the tree amplitudes vanish for any gravity theory and the loop amplitudes vanish for any supergravity theory. The first non-vanishing amplitudes are those with two negative helicity gravitons, known as “Maximally Helicity Violating” or MHV amplitudes. Amplitudes with three negative helicity gravitons are “next-to- MHV ” or $NMHV$ amplitudes. Amplitudes with exactly two positive helicity gravitons and the remaining negative helicity can be obtained by conjugation and are known alternatively as “googly amplitudes” or, as used by us, \overline{MHV} .

(The factors of κ have been reinstated in the amplitudes within this equation.) For a general amplitude the boxes with three or fewer massive legs, the one and two mass triangles and the bubble integrals all have $1/\epsilon$ singularities which can contribute to the above.

A necessary condition for the no-triangle hypothesis is that the box contributions alone yield the complete $1/\epsilon$ structure. In other words,

$$\sum_{i \in \mathcal{C}} c_i I_4^i \Big|_{1/\epsilon} = \frac{i}{(4\pi)^2} \left[\frac{\sum_{i < j} s_{ij} \ln[-s_{ij}]}{2\epsilon} \right] \times M_{(1,2,\dots,n)}^{\text{tree}}. \quad (3.2)$$

If this condition is satisfied, it implies the vanishing of a large number of the triangle coefficients, specifically that the one and two-mass triangle functions are not present. The one- and two-mass triangles are actually not an independent set of integral functions. As shown in the appendix they can be replaced by a set of basis functions,

$$G(-K^2) = \frac{(-K^2)^{-\epsilon}}{\epsilon^2} = \frac{1}{\epsilon^2} - \frac{\ln(-K^2)}{\epsilon} + \text{finite}, \quad (3.3)$$

where the set of G 's runs over all the independent momentum invariants, K^2 , of the amplitude. These functions plus the boxes then give the only $\ln(-K^2)/\epsilon$ contributions to the amplitude since the $1/\epsilon$ terms in bubbles do not contain logarithms. If the boxes completely reproduce the required singularity, the coefficients of the G functions must be zero and consequently the coefficients of the one- and two-mass triangles can be set to zero,

$$d_{1m,i} = d_{2m,i} = 0. \quad (3.4)$$

Having the correct soft behaviour only imposes a single constraint on the sum of the bubble coefficients,

$$\sum_i e_i = 0, \quad (3.5)$$

and, importantly, places no constraint on the three-mass triangles as they are IR finite.

To verify the IR behaviour, one must know the box coefficients. Fortunately, there has been considerable progress in computing the box coefficients in gauge theory. Box coefficients may be determined using unitarity [33, 34]. In ref. [35], Britto, Cachazo and Feng showed that quadruple cuts can be used to algebraically obtain box coefficients from the four tree amplitudes at the corners of the cut box. Specifically, if we consider an amplitude containing a scalar box integral function, the coefficient of this function is given by the product of four tree amplitudes with on-shell cut legs [35],

$$c = \frac{1}{2} \sum_{h_i \in \mathcal{S}} \left(M^{\text{tree}}((- \ell_1)^{-h_1}, i_1, \dots, i_2, (\ell_2)^{h_2}) \times M^{\text{tree}}((- \ell_2)^{-h_2}, i_3, \dots, i_4, (\ell_3)^{h_3}) \right. \\ \left. \times M^{\text{tree}}((- \ell_3)^{-h_3}, i_5, \dots, i_6, (\ell_4)^{h_4}) \times M^{\text{tree}}((- \ell_4)^{-h_4}, i_7, \dots, i_8, (\ell_1)^{h_1}) \right). \quad (3.6)$$

Here \mathcal{S} indicates the set of possible particle and helicity configurations of the legs ℓ_i which give a non-vanishing product of tree amplitudes. We often denote the above coefficient by the clustering on the legs, $c^{[\{i_1 \dots i_2\}, \{i_3 \dots i_4\}, \{i_5 \dots i_6\}, \{i_7 \dots i_8\}]}$. In the above the tree amplitudes at massless corners require analytic continuation.

The box coefficients may also be obtained from the known box coefficients for $\mathcal{N} = 4$ Yang-Mills [34, 18, 19] by squaring and summing [2]. For example for the three-mass boxes within the seven-point $NMHV$ amplitude we have,

$$c_{\mathcal{N}=8}^{[1^-, \{4^+ 5^+\}, \{2^- 3^-\}, \{6^+ 7^+\}]} = 2 s_{23} s_{45} s_{67} c_{\mathcal{N}=4}^{[1^-, \{4^+ 5^+\}, \{2^- 3^-\}, \{6^+ 7^+\}]} c_{\mathcal{N}=4}^{[1^-, \{5^+ 4^+\}, \{3^- 2^-\}, \{7^+ 6^+\}]} , \quad (3.7)$$

which allows us to obtain the $\mathcal{N} = 8$ coefficients from the $\mathcal{N} = 4$ box coefficients.

We have computed the IR behaviour of the six and seven-point $NMHV$ amplitudes. The six-point box coefficients are given in ref. [17] and the seven-point box coefficients are given in appendix A. In both cases amplitudes were constructed using these box-coefficients and, after some computer algebra, the resultant amplitudes were found to reproduce the complete IR behaviour. This allows us to conclude that,

$$d_{2m,i} = d_{1m,i} = 0 \text{ for } n = 6, 7. \quad (3.8)$$

3.2 Two-Particle Cuts

A general unitarity cut of the amplitude $M_n(1, 2, \dots, n)$ in the channel carrying momentum $P = k_i + \dots k_j$, is given by a sum of phase space integrals of products of tree amplitudes,

$$C_{i \dots j} = i \sum_{h_1, h_2 \in \mathcal{S}'} \int d\text{LIPS}(-l_1, l_2) M^{\text{tree}}((-l_1)^{-h_1}, i, \dots, j, (l_2)^{h_2}) \times M^{\text{tree}}((-l_2)^{-h_2}, j+1, \dots, i-1, (l_1)^{h_1}), \quad (3.9)$$

where \mathcal{S}' denotes the helicities of the particles from the $\mathcal{N} = 8$ -multiplet that can run in the loop. This unitarity cut is equal to the leading discontinuity of the loop amplitude,

$$\sum_{i \in \mathcal{C}} c_i I_4^i + \sum_{j \in \mathcal{D}} d_j I_3^j + \sum_{k \in \mathcal{E}} e_k I_2^k \Big|_{\text{Disc}} = i \int d\text{LIPS}(-l_1, l_2) \left[\sum_{i \in \mathcal{C}'} \frac{c_i}{(l_1 - K_{i,4})^2 (l_2 - K_{i,2})^2} + \sum_{j \in \mathcal{D}'} \frac{d_j}{(l_1 - K_{j,3})^2} + e_{k'} \right]. \quad (3.10)$$

The sets of box and triangle functions that contribute to a given cut are denoted by \mathcal{C}' and \mathcal{D}' respectively and the single bubble function that contributes is labelled by k' . In principle the coefficients of all the integral functions can be obtained by performing all possible two-particle cuts. In practice it is often simpler to determine the box and triangle coefficients by other means before using the two-particle cuts to determine the bubble terms. The rational pieces of the amplitude are not “cut-constructible” [33, 34].

To show that a given integral function is absent from the amplitude we have to show that its contribution to the cut integral vanishes. This test may be done by either evaluating the cut integral explicitly or, equivalently, by algebraically reducing the integrand to a sum of constant coefficients times specific products of propagators, that are the *signatures* of the cuts of specific integral functions.

3.3 Bubble Integrals from the two-particle cuts

In this section we will show, by explicit computation of the two-particle cuts, that all bubble integrals in the six-point amplitudes vanish. These arguments can also be used to show that bubble integrals are absent from all the cuts of all-n one-loop *MHV* amplitudes as discussed in section (4.2).

Recently, the realisation that Yang-Mills amplitudes are dual to a twistor string theory [36] has given considerable impetus to gauge theory calculations. In particular, it appears that the two-particle cuts can be efficiently calculated if expressed in spinor or twistor variables [30].

Consider the two-particle cut,

$$C_{12} = i \int d\mu M_4^{\text{tree}}((-l_1)^+, 1^-, 2^-, (l_2)^+) \times M_6^{\text{tree}}((l_1)^-, 3^-, 4^+, 5^+, 6^+, (-l_2)^-), \quad (3.11)$$

where $d\mu = d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^{(4)}(l_1 - l_2 - k_1 - k_2)$,

with a graviton running in the loop. We denote the integrand by $\mathcal{I}(l_1, l_2)$. Setting $l_1 = t\ell$ with $\ell_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, the measure becomes [37, 30],

$$d^4 l_1 \delta^{(+)}(l_1^2) = t dt \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] , \quad (3.12)$$

so that the cut becomes,

$$\begin{aligned} C_{12} &= i \int d\mu \mathcal{I}(l_1, l_2) = i \int_0^\infty t dt \int \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \delta^{(+)}(P^2 - t P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}}) \mathcal{I}(t\ell, -P - t\ell) \\ &= i \int \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \frac{P^2}{(P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})^2} \mathcal{I}\left(\frac{P^2}{P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}}} \ell, -P - \frac{P^2}{P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}}} \ell\right), \end{aligned} \quad (3.13)$$

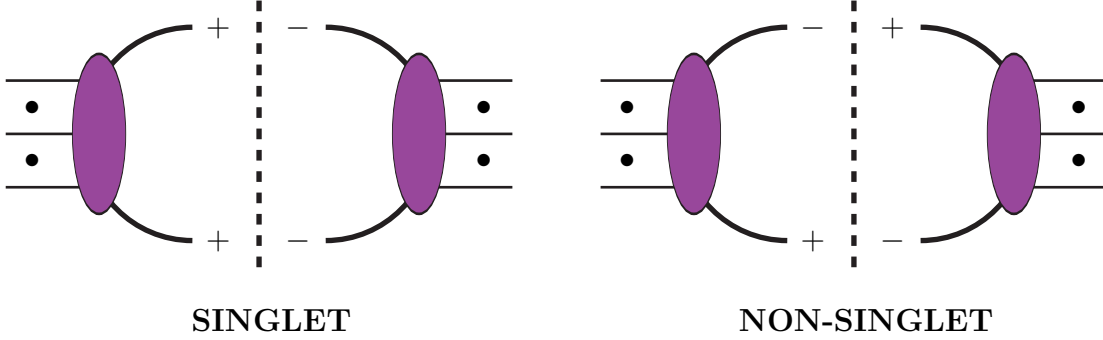
where P denotes the total momentum on one side of the cut. In the example above, $P = k_1 + k_2$. Powers of l_1 within \mathcal{I} give rise to powers of t which in turn give rise to extra powers of $P^2/(P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})$ due to the $\delta(P^2 - t P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})$ -function. Thus in general the cut will be a sum of terms with different powers of $(P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})$,

$$C_{12} = \int \langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}] \sum_n \frac{f_n(\lambda, \tilde{\lambda})}{(P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})^n}. \quad (3.14)$$

The key observation of [30, 38] is that the different classes of integral function that contribute to the cut can be recognised by the powers of $(P_{a\dot{a}} \lambda^a \tilde{\lambda}^{\dot{a}})$ that are present.

Generically, any term containing, $1/(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n$ with $n < 2$ in C_{12} will not generate a contribution to the coefficient of any bubble integral function. In terms of t , such terms correspond to terms in \mathcal{I} of the form t^m with $m < 1$. In the following we show that only terms of this form arise in two-particle cuts of the six-point one-loop amplitudes and hence that no bubble integral functions contribute to these amplitudes.

The $NMHV$ amplitude $M^{1\text{-loop}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ has four inequivalent cuts up to relabelling of external legs; C_{12} , C_{34} , C_{123} and C_{234} . Of these C_{12} and C_{123} are what we call singlet cuts. These cuts vanish unless the two outgoing cut legs have the same helicity, implying that these states can only be gravitons. These singlet cuts are thus independent of the matter content of the theory and the absence of bubble functions is independent of the number of supersymmetries. For the non-singlet cuts, the two outgoing cut legs have opposite helicity and so the full $\mathcal{N} = 8$ multiplet contributes. For these cuts, bubble functions are only absent from the $\mathcal{N} = 8$ amplitudes.



We now examine the four distinct cuts in turn. First we consider C_{123} , as this is the simplest: it is a singlet cut and the tree amplitudes that appear are either MHV or \overline{MHV} amplitudes. Explicitly the product of tree amplitudes is,

$$\begin{aligned}
M_{\text{MHV}}^{\text{tree}}((-l_1)^+, 1^-, 2^-, 3^-, (l_2)^+) &\times M_{\text{MHV}}^{\text{tree}}((-l_2)^-, 4^+, 5^+, 6^+, (l_1)^-) \\
&= -[l_1 l_2]^8 \frac{[3 l_1] \langle l_1 1 \rangle [1 2] \langle 2 3 \rangle - \langle 3 l_1 \rangle [l_1 1] \langle 1 2 \rangle [2 3]}{[l_1 l_2] [l_1 1] [l_1 2] [l_1 3] [l_2 1] [l_2 2] [l_2 3] [1 2] [1 3] [2 3]} \\
&\times \langle l_1 l_2 \rangle^8 \frac{\langle 6 l_1 \rangle [l_1 4] \langle 4 5 \rangle [5 6] - [6 l_1] \langle l_1 4 \rangle [4 5] \langle 5 6 \rangle}{\langle l_1 l_2 \rangle \langle l_1 4 \rangle \langle l_1 5 \rangle \langle l_1 6 \rangle \langle l_2 4 \rangle \langle l_2 5 \rangle \langle l_2 6 \rangle \langle 4 5 \rangle \langle 4 6 \rangle \langle 5 6 \rangle}.
\end{aligned} \tag{3.15}$$

This can be simplified to,

$$\begin{aligned}
&= - \frac{(P_{123}^2)^{10}}{[1 2] [1 3] [2 3] \langle 4 5 \rangle \langle 4 6 \rangle \langle 5 6 \rangle} \times \\
&\frac{([3 l_1] \langle l_1 1 \rangle [1 2] \langle 2 3 \rangle - \langle 3 l_1 \rangle [l_1 1] \langle 1 2 \rangle [2 3]) (\langle 6 l_1 \rangle [l_1 4] \langle 4 5 \rangle [5 6] - [6 l_1] \langle l_1 4 \rangle [4 5] \langle 5 6 \rangle)}{[l_1 1] [l_1 2] [l_1 3] \langle l_1 4 \rangle \langle l_1 5 \rangle \langle l_1 6 \rangle \prod_{x=1,2,3} \langle l_1 | P_{123} | x \rangle \prod_{y=4,5,6} [l_1 | P_{123} | y]}
\end{aligned} \tag{3.16}$$

Substituting $l_1 = tl$ into the above term we find a factor of $1/t^4$ and hence there are no bubble contributions to this cut.

Next we consider C_{234} . Again the tree amplitudes are either MHV or \overline{MHV} amplitudes, but this is a non-singlet cut, so we must include a summation over the super-multiplet. $MHV(\overline{MHV})$ tree amplitudes with a single pair of non-graviton particles are related to the corresponding pure graviton amplitude by simple factors, $X(h)$. The summed integrand is most naturally expressed in terms of tree amplitudes with a scalar circulating in the loop and a ρ -factor. Using a superscript s to denote a scalar in the loop, we have,

$$\begin{aligned} \sum_{h \in S'} M^{\text{tree}}((-l_1)^{-h}, 2^-, 3^-, 4^+, (l_2)^h) \times M^{\text{tree}}((-l_2)^{-h}, 5^+, 6^+, 1^-, (l_1)^h) \\ = M^{\text{tree}}((-l_1)^s, 2^-, 3^-, 4^+, (l_2)^s) \times M^{\text{tree}}((-l_2)^s, 5^+, 6^+, 1^-, (l_1)^s) \sum_{h \in S'} X(h) \quad (3.17) \\ = \rho \times M^{\text{tree}}((-l_1)^s, 2^-, 3^-, 4^+, (l_2)^s) \times M^{\text{tree}}((-l_2)^s, 5^+, 6^+, 1^-, (l_1)^s), \end{aligned}$$

where,

$$\rho = \sum_{h \in S'} X(h) = \sum_{a=-4}^{a=4} \frac{8!}{(4-a)!(4+a)!} \left(\frac{x}{y}\right)^a = \frac{(x+y)^8}{x^4 y^4} = \frac{\langle 1 | P_{234} | 4 \rangle^8}{([4 l_1] [4 l_2] \langle 1 l_1 \rangle \langle 1 l_2 \rangle)^4}. \quad (3.18)$$

The factor $n_h = 8!/((4-a)!(4+a)!)$ is the multiplicity within the $\mathcal{N} = 8$ multiplet of the states of helicity $h = a/2$. Rewriting the amplitude in terms of l_1 we can count the powers of t . Overall the leading contributions are $O(t^{-4})$, just as in the singlet case. Once again the cut receives no contributions from bubble functions.

The remaining cuts are algebraically more complicated, but they repeat the patterns seen above. C_{12} is a singlet cut involving the product of a four-point MHV amplitude, $M^{\text{tree}}((-l_1)^+, 1^-, 2^-, (l_2)^+)$, and a six-point $NMHV$ amplitude, $M^{\text{tree}}((-l_2)^-, 3^-, 4^+, 5^+, 6^+, (l_1)^-)$. The six-point $NMHV$ tree amplitude has only recently been calculated using on-shell recursion [39–41]. An explicit form for this amplitude as a sum of fourteen terms is given in appendix C.³

³In general much less is known about gravity tree amplitudes than Yang-Mills amplitudes. Traditional Feynman diagram approaches tend to be excessively complicated as evidenced by the computation by Sannon [8] of the four-point tree amplitude. The KLT relations, which express the gravity amplitudes as sums of permutations of products of two Yang-Mills amplitudes [9], are an extremely useful technique, however the factorisation structure is rather obscure and the permutation sum grows quickly with the number of legs. Of the new techniques, the BCF recursion readily extends to gravity amplitudes [40, 41] giving useful compact results. The MHV -vertex approach of Cachazo, Svrček and Witten also extends to gravity [42] although the correct analytic continuation of the MHV gravity vertices is only clear after using the appropriate factorisation [43]. Currently, there is no Lagrangian based proof of these techniques such as exists for Yang-Mills [44], however we have numerically checked the expressions for both MHV vertices and recursion against the KLT expressions for amplitudes with seven or fewer points.

We will illustrate here how one of the terms gives a contribution that vanishes at large t . The remaining thirteen terms will follow analogously and thus we see term-by-term that this cut receives no contributions from bubble functions. The singlet 12-cut reads,

$$C_{12} = i \int d\mu M_4((-l_1)^+, 1^-, 2^-, (l_2)^+) \times M_6((-l_2)^-, 3^-, 4^+, 5^+, 6^+, (l_1)^-) , \quad (3.19)$$

where the four-point amplitude is,

$$M_4((-l_1)^+, 1^-, 2^-, (l_2)^+) = \frac{i \langle 1\ 2 \rangle^7 [1\ 2]}{\langle 1\ l_1 \rangle \langle 1\ l_2 \rangle \langle 2\ l_1 \rangle \langle 2\ l_2 \rangle \langle l_1\ l_2 \rangle^2} , \quad (3.20)$$

and the six-point amplitude is given in [17]. We will in this example analyse the contribution to the cut given by the term $G_4^{ns}[-l_2, 3, 5, 4, 6, l_1]$ in the full amplitude,

$$M_6((-l_2)^-, 3^-, 4^+, 5^+, 6^+, (l_1)^-) |_{G_4^{ns}[-l_2, 3, 5, 4, 6, l_1]} = \frac{i s_{35} s_{46} s_{l_1 l_2} [5 | P_{l_2 35} | l_1 \rangle^7}{\langle 46 \rangle^2 \langle 4l_1 \rangle \langle 6l_1 \rangle [35]^2 [3l_2] [5l_2] [3 | P_{l_2 35} | l_1 \rangle [l_2 | P_{l_2 35} | 4] [l_2 | P_{l_2 35} | 6] t_{35 l_2}} , \quad (3.21)$$

so that the integrand of (3.19) is,

$$\frac{-\langle 12 \rangle^7 [12]}{\langle 1l_1 \rangle \langle 1l_2 \rangle \langle 2l_1 \rangle \langle 2l_2 \rangle \langle l_1 l_2 \rangle^2} \times \frac{s_{35} s_{46} s_{l_1 l_2} [5 | P_{l_2 35} | l_1 \rangle^7}{\langle 46 \rangle^2 \langle 4l_1 \rangle \langle 6l_1 \rangle [35]^2 [3l_2] [5l_2] [3 | P_{l_2 35} | l_1 \rangle [l_2 | P_{l_2 35} | 4] [l_2 | P_{l_2 35} | 6] t_{35 l_2}} , \quad (3.22)$$

which can be written as,

$$\mathcal{C} \times \frac{[5 | P_{l_2 35} | l_1 \rangle^7}{\langle 1l_2 \rangle \langle 1l_1 \rangle \langle 2l_2 \rangle \langle 2l_1 \rangle \langle l_2 l_1 \rangle^2 \langle 4l_1 \rangle \langle 6l_1 \rangle [3l_2] [5l_2] [3 | P_{l_2 35} | l_1 \rangle [l_2 | P_{l_2 35} | 4] [l_2 | P_{l_2 35} | 6] t_{35 l_2}} , \quad (3.23)$$

where,

$$\mathcal{C} = \frac{s_{35} s_{46} s_{12}^2 \langle 1\ 2 \rangle^6}{\langle 46 \rangle^2 [35]^2} . \quad (3.24)$$

Now transforming all l_2 into l_1 using $[X l_2] \rightarrow \frac{[X | P_{12} | l_1 \rangle}{\langle l_2 l_1 \rangle}$ and $\langle Y l_2 \rangle \rightarrow \frac{\langle Y | P_{12} | l_1 \rangle}{[l_2 l_1]}$ we get,

$$\frac{s_{12}^2 \mathcal{C}}{\langle 1\ 2 \rangle^2} \times \mathcal{H}(|l_1 \rangle) \times \frac{1}{[2 l_1] [1 l_1] t_{46 l_1}} , \quad (3.25)$$

where,

$$\mathcal{H}(|l_1 \rangle) = \frac{[5 | P_{46} | l_1 \rangle^7}{\langle 1l_1 \rangle \langle 2l_1 \rangle \langle 4l_1 \rangle \langle 6l_1 \rangle [3 | P_{12} | l_1 \rangle [5 | P_{12} | l_1 \rangle [3 | P_{46} | l_1 \rangle \langle l_1 | P_{12} P_{46} | 4] \langle l_1 | P_{12} P_{46} | 6] .$$

Now we have to count the number of factors of t . We get a total count of $1/t^2$ hence no bubbles integral functions are present in the cut.

The remaining C_{34} cut is non-singlet and so we again need to sum over the multiplet. Explicit forms for the relevant six-point amplitudes involving an arbitrary pair of particles plus gravitons are given in appendix C. These tree amplitudes are each a sum of fourteen terms. As we change the non-graviton particles, the individually terms in the amplitude each behave like MHV amplitudes in that they collect simple multiplicative factors. Performing the sum over the multiplet term-by-term we find a ρ -factor for each term. Just as in the C_{234} cut, these are very important as they introduce large inverse powers of t . For most terms, $\rho \sim 1/t^8$. Again we pick a sample term to illustrate the process: the other thirteen terms follow analogously.

We will consider the cut,

$$C_{34} = i \int d\mu \sum_{h \in \mathcal{S}'} M_4((-l_1)^h, 3^-, 4^+, (l_2)^{-h}) M_6((-l_2)^h, 5^+, 6^+, 1^-, 2^-, (l_1)^{-h}). \quad (3.26)$$

The four-point tree amplitude $M_4((-l_1)^h, 3^-, 4^+, (l_2)^{-h})$ is given by,

$$M_4((-l_1)^h, 3^-, 4^+, (l_2)^{-h}) = \frac{i \langle l_2 3 \rangle^7 [l_2 3]}{\langle 34 \rangle \langle 3l_1 \rangle \langle 4l_1 \rangle^2 \langle l_2 4 \rangle \langle l_2 l_1 \rangle} \left(\frac{\langle -l_1 3 \rangle}{\langle l_2 3 \rangle} \right)^{4-2h}. \quad (3.27)$$

For the six-point corner we consider a specific but representative term from the fourteen in eq (C.2),

$$\begin{aligned} & M_6((-l_2)^h, 5^+, 6^+, 1^-, 2^-, (l_1)^{-h})|_{T_2} \\ &= \left(-\frac{i \langle 1l_2 \rangle [6l_1]}{\langle 6|P_{26l_1}|1 \rangle} \right)^{4-2h} \times \\ & \quad \frac{-i \langle 2l_1 \rangle \langle 1|P_{26l_1}|6 \rangle^8 [5l_2]}{\langle 15 \rangle \langle 1l_2 \rangle \langle 5l_2 \rangle \langle 1|P_{26l_1}|2 \rangle \langle 1|P_{26l_1}|l_1 \rangle \langle 5|P_{26l_1}|6 \rangle \langle l_2|P_{26l_1}|6 \rangle [26] [2l_1] [6l_1] t_{26l_1}}. \end{aligned} \quad (3.28)$$

The particle type dependent factors can be extracted and we find relative to the graviton amplitude,

$$\rho = \sum_{h \in \mathcal{S}'} \left(-\frac{i \langle 1l_2 \rangle [6l_1] \langle -l_1 3 \rangle}{[6|P_{26l_1}|1 \rangle \langle l_2 3 \rangle} \right)^{4-2h} = \left(\frac{-\langle 12 \rangle [26] \langle 3l_2 \rangle + \langle 13 \rangle [6|P_{34}|l_2 \rangle]}{[6|P_{15-l_2}|1 \rangle \langle l_2 3 \rangle} \right)^8. \quad (3.29)$$

Next we rewrite the cut in terms of the loop momenta l_2 using the on-shell conditions and $l_1 = l_2 + k_3 + k_4$. The ρ factor already has the correct form. The remaining contributions to the cut integral are then,

$$\mathcal{C} \times \mathcal{H}(|l_2 \rangle) \times \frac{\langle 2|P_{34}|l_2 \rangle [5l_2]}{[3l_2] [4l_2] \langle 1|P_{15-l_2}|2 \rangle \langle 5|P_{15-l_2}|6 \rangle t_{15-l_2}}, \quad (3.30)$$

where

$$\mathcal{C} = \frac{[34]^2}{\langle 34 \rangle^2 \langle 15 \rangle [26]}, \quad (3.31)$$

and

$$\mathcal{H}(|l_2\rangle) = \frac{(-\langle 12 \rangle [26] \langle 3l_2 \rangle + \langle 13 \rangle [6|P_{34}|l_2\rangle])^8}{\langle l_2 3 \rangle \langle l_2 4 \rangle \langle 1l_2 \rangle \langle 5l_2 \rangle [6|P_{34}|l_2] [2|P_{34}|l_2] \langle l_2|P_{15}|6 \rangle \langle 1|P_{26}P_{34}|l_2 \rangle}. \quad (3.32)$$

We now replace l_2 by $l_2 = t\ell = t\lambda\tilde{\lambda}$ and do the t -integration. With the definitions, $\langle \ell|Q_1|\ell \rangle = \langle \ell|P_{34}|\ell \rangle \langle 1|P_{15}|2 \rangle - s_{34} \langle \ell 1 \rangle [2\ell]$, $\langle \ell|Q_2|\ell \rangle = \langle \ell|P_{34}|\ell \rangle \langle 5|P_{15}|6 \rangle - s_{34} \langle \ell 5 \rangle [6\ell]$, $\langle \ell|Q_3|\ell \rangle = \langle \ell|s_{15}P_{34} - s_{34}P_{15}|\ell \rangle$, we can rewrite the cut as,

$$-\mathcal{C} \times s_{34} \times \mathcal{H}(|l\rangle) \times \langle \ell|P_{34}|\ell \rangle \times \frac{i \langle 2|P_{34}|\ell \rangle [5\ell]}{[3\ell] [4\ell] \langle \ell|Q_1|\ell \rangle \langle \ell|Q_2|\ell \rangle \langle \ell|Q_3|\ell \rangle}. \quad (3.33)$$

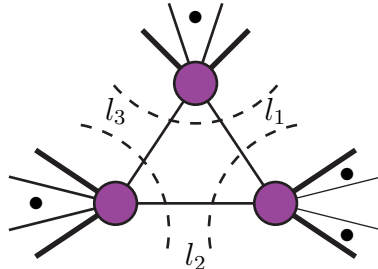
It is important to notice that the ρ -factor contributes $\langle \ell|P_{34}|\ell \rangle^8$, while the product of the graviton amplitudes gives rise to $1/\langle \ell|P_{34}|\ell \rangle^5$, with a further factor of $1/\langle \ell|P_{34}|\ell \rangle^2$ coming from the integration measure. The powers of $\langle \ell|P_{34}|\ell \rangle$ are important in that they indicate the type of integral functions that are present. For the above term with only single poles in the denominator, bubbles can only arise from terms carrying a factor of $1/\langle \ell|P_{34}|\ell \rangle^2$. We therefore conclude that no bubbles are present.

By considering all distinct two-particle cuts of the six-point one-loop $NMHV$ amplitude we have shown that the amplitude receives no contributions from bubble integral functions.

3.4 Triple Cuts

Having verified that no one or two-mass triangles or bubble integral functions are present in the amplitude, we now consider the three-mass triangle integral function. These have no IR singularities and so the previous arguments have nothing to say regarding their absence or presence. In this section we illustrate how the coefficients of three-mass triangles can be evaluated by numerically integrating triple cuts of amplitudes. Note that MHV amplitudes do not contain triple cuts for any gravity theory so this is a previously untested class of functions.

Consider a physical triple cut in an amplitude where all three corners are massive,



$$C_3 = \sum_{h_i \in \mathcal{S}'} \int d^4 l_1 \delta(l_1^2) \delta(l_2^2) \delta(l_3^2) M((l_1)^{h_1}, i_m, \dots, i_j, (-l_2)^{-h_2}) \times M((l_2)^{h_2}, i_{j+1}, \dots, i_l, (-l_3)^{-h_3}) \times M((l_3)^{h_3}, i_{l+1}, \dots, i_{m-1}, (-l_1)^{-h_1}), \quad (3.34)$$

where the summation is over all possible intermediate states. As the momentum invariants, $K_1 = k_{i_m} + k_{i_m+1} + \dots + k_{i_j}$ etc, are all non-null, there exist kinematic regimes in which the integration has non-vanishing support for real loop momentum. In such cases the remaining one dimensional integral can readily be evaluated numerically. In the generic expression of an amplitude the only integral functions contributing to the triple cut are box functions and the specific three mass triangle for the cut,

$$C_3 = \sum_i c_i (I_4^i)_{\text{triple-cut}} + d_{3m} (I_3^{3m})_{\text{triple-cut}}. \quad (3.35)$$

The box functions which can contribute are the two-mass-hard, three-mass and four mass. This equation can be inverted to express the coefficient d_{3m} in terms of C_3 and the box-coefficients.

For the six-point case the cut,

$$C_3 = \sum_{h_i \in \mathcal{S}'} \int d^4 l_i \delta(l_1^2) \delta(l_2^2) \delta(l_3^2) M_4((l_1)^{h_1}, 1, 2, (-l_2)^{-h_2}) \\ \times M_4((l_2)^{h_2}, 3, 4, (-l_3)^{-h_3}) \times M_4((l_3)^{h_3}, 5, 6, (-l_1)^{-h_1}), \quad (3.36)$$

only receives contributions from two-mass-hard boxes, such as $I_4^{2m\ h}[2, \{3, 4\}, \{5, 6\}, 1]$, and the three mass triangle. The triple cut of a two-mass hard box is,

$$(I_4^{2m\ h})_{\text{triple-cut}} = \int \frac{d^4 p}{p^2(p-k_2)^2(p-k_2-K_3)^2(p+k_1)^2} \Big|_{\text{cut}} \\ = \int \frac{d^4 p \delta((p-k_2)^2) \delta((p-k_2-K_3)^2) \delta((p+k_1)^2)}{p^2} \\ = \frac{\pi}{2(k_1+k_2)^2(k_2+K_3)^2}, \quad (3.37)$$

while the triple cut of the three mass triangle is,

$$(I_3^{3m})_{\text{triple-cut}} = \int \frac{d^4 p}{p^2(p-K_1)^2(p+K_3)^2} \Big|_{\text{cut}} \\ = \int \frac{d^4 p \delta(p^2) \delta((p+K_3)^2) \delta((p-K_1)^2)}{\pi} = \frac{\pi}{2\sqrt{\Delta_3}} \\ = \frac{\pi}{2\sqrt{(K_1^2)^2 + (K_2^2)^2 + (K_3^2)^2 - 2(K_1^2 K_2^2 + K_1^2 K_3^2 + K_2^2 K_3^2)}}. \quad (3.38)$$

Thus we see that,

$$\frac{\pi}{2\sqrt{\Delta_3}} d_3^{3m} = C_3 - \frac{\pi}{2} \sum_i \frac{c_{2m\ h,i}}{(k_1+k_2)^2(k_2+K_3)^2}. \quad (3.39)$$

The integral in C_3 is well behaved and can be determined numerically from the tree amplitudes. Using the box-coefficients for the six-point amplitude [17] we have verified numerically that,

$$d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+\}, \{5^+, 6^+\}] = 0, \\ d_3^{3m}[\{1^-, 4^+\}, \{2^-, 5^+\}, \{3^-, 6^+\}] = 0. \quad (3.40)$$

The first zero is true for any (massless) gravity theory whilst the second is true only for $\mathcal{N} = 8$ supergravity.

For the seven-point amplitude we must also include three-mass boxes in the triple cut. Using the seven-point box coefficients given in the appendix we have verified that three mass triangles are absent in the seven-point *NMHV* amplitude. Explicitly,

$$\begin{aligned} d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+\}, \{5^+, 6^+, 7^+\}] &= 0, \\ d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+, 5^+\}, \{6^+, 7^+\}] &= 0, \\ d_3^{3m}[\{1^-, 2^-, 4^+\}, \{3^-, 5^+\}, \{6^+, 7^+\}] &= 0, \\ d_3^{3m}[\{1^-, 4^+\}, \{2^-, 5^+\}, \{3^-, 6^+, 7^+\}] &= 0, \end{aligned} \tag{3.41}$$

with the first three coefficients vanishing for any matter content but the last only zero for $\mathcal{N} = 8$ supergravity.

3.5 Factorisation

The unitarity constraints of the previous sections are sufficient to show the absence of integral functions involving logarithms. This is sufficient to prove the no-triangle hypothesis for six or fewer gravitons. At seven-point and beyond, the amplitude may, in principle, contain rational terms which do not appear in the four-dimensional cuts. Unitarity can be used to obtain these [45–48] but one must evaluate the cuts fully in $4 - 2\epsilon$ dimensions. Recently, there has been much progress in determining the rational parts of QCD one-loop amplitudes based on the physical factorisations of the amplitudes [49–51]. Gravity amplitudes are also heavily constrained by factorisation so the absence of terms other than boxes for six or fewer legs makes it difficult to envisage their presence at higher points.

More explicitly, consider the multi-particle factorisations. From general field theory considerations, amplitudes must factorise (up to subtleties having to do with infrared singularities) on multi-particle poles. For $K^\mu \equiv k_i^\mu + \dots + k_{i+r+1}^\mu$ the amplitude factorises when K becomes on shell. Specifically, as $K^2 \rightarrow 0$ the factorisation properties of one-loop massless amplitudes are described by [52],

$$\begin{aligned} M_n^{1\text{-loop}} \xrightarrow{K^2 \rightarrow 0} & \sum_{\lambda=\pm} \left[M_{r+1}^{1\text{-loop}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} M_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \right. \\ & + M_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} M_{n-r+1}^{1\text{-loop}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \\ & \left. + M_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} M_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \hat{r}_\Gamma \mathcal{F}_n(K^2; k_1, \dots, k_n) \right], \end{aligned} \tag{3.42}$$

where the one-loop “factorisation function” \mathcal{F}_n is helicity independent.

Gravity one-loop amplitudes also have soft and collinear factorisations. In ref. [16] it was shown that these have a universal collinear behaviour given by,

$$M_n^{1\text{-loop}}(\dots, a^{\lambda_a}, b^{\lambda_b}, \dots) \xrightarrow{a \parallel b} \sum_{\lambda} \text{Split}^{\text{gravity}}(z, a^{\lambda_a}, b^{\lambda_b}) \times M_{n-1}^{1\text{-loop}}(\dots, P^{\lambda}, \dots), \quad (3.43)$$

when k_a and k_b are collinear. The pure graviton splitting amplitudes are,

$$\begin{aligned} \text{Split}^{\text{gravity}}_+(z, a^+, b^+) &= 0, \\ \text{Split}^{\text{gravity}}_-(z, a^+, b^+) &= -\frac{1}{z(1-z)} \frac{[a b]}{\langle a b \rangle}, \\ \text{Split}^{\text{gravity}}_+(z, a^-, b^+) &= -\frac{z^3}{1-z} \frac{[a b]}{\langle a b \rangle}. \end{aligned} \quad (3.44)$$

There is also a universal soft behaviour given by,

$$M_n^{1\text{-loop}}(\dots, a, s^{\pm}, b, \dots) \xrightarrow{k_s \rightarrow 0} \mathcal{S}^{\text{gravity}}(s^{\pm}) \times M_{n-1}^{1\text{-loop}}(\dots, a, b, \dots), \quad (3.45)$$

when k_s becomes soft. For the limit $k_n \rightarrow 0$ in $M_n^{\text{tree}}(1, 2, \dots, n)$, the gravitational soft factor (for positive helicity) is,

$$\mathcal{S}_n \equiv \mathcal{S}^{\text{gravity}}(n^+) = \frac{-1}{\langle 1 n \rangle \langle n, n-1 \rangle} \sum_{i=2}^{n-2} \frac{\langle 1 i \rangle \langle i, n-1 \rangle [i n]}{\langle i n \rangle}. \quad (3.46)$$

Note that the collinear behaviour is only a “phase singularity” for real momenta [16], however it should be regarded as a genuine singularity when using complex momenta.

These factorisations place constraints on the rational terms R_n . Since $R_n = 0$ for $n \leq 6$ the natural solution is $R_n = 0$ for all n . For QCD the factorisation constraints have been turned into recursion relations for the rational terms [49, 50]. If this bootstrap also applies to gravity amplitudes then we would be able to immediately deduce that $R_n = 0$ for $\mathcal{N} = 8$ amplitudes. At present a direct calculation of the rational terms beyond six-points seems unfeasible although there has been recent progress in producing algorithms focused on computing the rational terms in six-point QCD amplitudes [53, 54].

4. Checking Bubble-cuts by Large- z Shifts

In this section we look at a different way to test for bubble integral functions in the two-particle cuts. This approach is based on the scaling behaviour of amplitudes under specific shifts of the loop momenta.

Starting from equations (3.9) and (3.10), lifting the integral implies,

$$\begin{aligned} &M^{\text{tree}}((-l_1)^-, i, \dots, j, (l_2)^-) \times M^{\text{tree}}((-l_2)^+, j+1, \dots, i-1, (l_1)^+) \\ &= \sum_{i \in \mathcal{C}'} \frac{c_i}{(l_1 - K_{i,4})^2 (l_2 - K_{i,2})^2} + \sum_{j \in \mathcal{D}'} \frac{d_j}{(l_1 - K_{j,3})^2} + e_{k'} + D(l_1, l_2). \end{aligned} \quad (4.1)$$

Here $D(l_1, l_2)$ is a total derivative, $\int d\text{LIPS}(-l_1, l_2)D(l_1, l_2) = 0$, which may or may not be present. Note that in the above a number of boxes and triangles may contribute but only one bubble. Let us consider (4.1) under the shift of the two-cut legs,

$$\lambda_{l_1} \longrightarrow \lambda_{l_1} + z \lambda_{l_2}, \quad \tilde{\lambda}_{l_2} \longrightarrow \tilde{\lambda}_{l_2} - z \tilde{\lambda}_{l_1}. \quad (4.2)$$

This shift does not change the coefficients but it does enter the propagator terms (and possibly the $D(l_1, l_2)$). In the large- z limit the propagators will vanish as $1/z$ leaving behind the bubble coefficient $e_{k'}$. This suggests a test for bubble terms: if,

$$\lim_{z \rightarrow \infty} M^{\text{tree}}((-l_1)^{-h_1}, i, \dots, j, (l_2)^{h_2}) \times M^{\text{tree}}((-l_2)^{-h_2}, j+1, \dots, i-1, (l_1)^{h_1}) \longrightarrow 0, \quad (4.3)$$

in the large- z limit, then,

$$e_{k'} = 0, \quad (4.4)$$

under the assumption that the total derivative vanishes at infinity. In the following section we discuss criteria for when this test may be used. This test is particularly useful as in many cases it follows from the general behaviour of gravity tree amplitudes and may be tested numerically when the tree amplitudes are known.

4.1 Relation to large t behaviour

A key step is to relate the large z behaviour to the large t behaviour of the cut parameterised as in the previous section. In that section, following [30, 38], we discussed how the integral functions that a given term in a unitarity cut contributes to are determined by the power, n , of $1/(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n$. A term in the cut integral (3.13) can be written as a rational expression in holomorphic and anti-holomorphic spinors λ^a and $\tilde{\lambda}^{\dot{a}}$ respectively (recall these spinors are NOT the same as the λ_{l_i} but are related via $l_1 = t\lambda\tilde{\lambda}$),

$$\int \frac{\langle \lambda d\lambda \rangle [\tilde{\lambda} d\tilde{\lambda}]}{(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n} \frac{n_{a_1\dots a_j, \dot{a}_1\dots \dot{a}_k} \lambda^{a_1} \dots \lambda^{a_j} \tilde{\lambda}^{\dot{a}_1} \dots \tilde{\lambda}^{\dot{a}_k}}{d_{b_1\dots b_l, \dot{b}_1\dots \dot{b}_m} \lambda^{b_1} \dots \lambda^{b_l} \tilde{\lambda}^{\dot{b}_1} \dots \tilde{\lambda}^{\dot{b}_m}}, \quad (4.5)$$

where the tensors $n_{a_1\dots a_j, \dot{a}_1\dots \dot{a}_k}$ and $d_{b_1\dots b_l, \dot{b}_1\dots \dot{b}_m}$ contain no factors of $(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})$.

Since the integrand must carry spinor weight -2 in λ and $\tilde{\lambda}$, the counters j, k, l, m and n obey $j - l - n = -2$ and $k - m - n = -2$. The $n_{a_1\dots \dot{a}_k}$ are non-vanishing for the contractions,

$$n_{a_1\dots a_j}^{\dot{a}_1\dots \dot{a}_k} \lambda^{a_1} \dots \lambda^{a_j} (\lambda^{b_1} P_{b_1\dot{a}_1}) \dots (\lambda^{b_k} P_{b_k\dot{a}_k}) \neq 0, \quad (4.6)$$

and similarly for $d_{b_1\dots \dot{b}_m}$. This can always be achieved: were the above contraction to vanish for all values of λ , then the spinor obtained by contracting all but one index has to be parallel to $\lambda^{b_k} P_{b_k}^{\dot{a}_k}$, that is,

$$n_{a_1\dots a_j}^{\dot{a}_1\dots \dot{a}_k} \lambda^{a_1} \dots \lambda^{a_j} (\lambda^{b_1} P_{b_1\dot{a}_1}) \dots (\lambda^{b_{k-1}} P_{b_{k-1}\dot{a}_{k-1}}) = n'(\lambda, P\lambda) (\lambda^{b_k} P_{b_k}^{\dot{a}_k}), \quad (4.7)$$

with n' a tensor of lower rank in λ and $\tilde{\lambda}$. We would then be able to pull out a power of $(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})$ and write the contraction of n as,

$$n(\lambda, \tilde{\lambda}) = n'(\lambda, \tilde{\lambda}) (P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}}), \quad (4.8)$$

contrary to our condition that no such factors exist.

The central observation is that the power n of $1/(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n$ is related to the leading power in large- z of λ_{l_i} . The shift in (4.2) translates into a shift on the λ and $\tilde{\lambda}$ of,

$$\lambda_a \longrightarrow \lambda_a, \quad \tilde{\lambda}_{\dot{a}} \rightarrow \tilde{\lambda}_{\dot{a}} + z \lambda^a P_{a\dot{a}}. \quad (4.9)$$

The terms $(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})$ are invariant under the shift, so the leading term at large- z is given by,

$$z^{k-m} \frac{1}{(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n} \times \frac{n(\lambda, P\lambda)}{d(\lambda, P\lambda)}. \quad (4.10)$$

Using $n = (k - m) + 2$ one finds that the large- z scaling is,

$$\sim z^{n-2}, \quad (4.11)$$

for a term with a $1/(P_{a\dot{a}}\lambda^a\tilde{\lambda}^{\dot{a}})^n$ factor. Consequently, if the product of the two tree amplitudes vanishes as $z \longrightarrow \infty$ then this product can only be composed of terms with $n \leq 1$. These terms do not contribute to bubble functions and hence the coefficient of the bubble corresponding to this cut must vanish.⁴

4.2 Using the large- z test for Gravity Amplitudes.

In this section we apply the test of the previous section to the two-particle cuts for graviton scattering in $\mathcal{N} = 8$ supergravity. We can use the behaviour of the gravity amplitudes under the shift (4.2) to determine the behaviour of the cut. We will need to consider two types of cut: singlet cuts where only graviton amplitudes are needed and non-singlet cuts where amplitudes involving other states in the supergravity multiplet contribute.

It is useful to briefly review the known results for the large- z behaviour of gravity amplitudes under the shifts,

$$\lambda_i \rightarrow \lambda_i + z \lambda_j, \quad \tilde{\lambda}_j \rightarrow \tilde{\lambda}_j - z \tilde{\lambda}_i. \quad (4.12)$$

The scaling of a given tree amplitude depends on the helicity of the two shifted legs and the helicity of the scattering gravitons. For MHV -amplitudes there is an

⁴Note that since there is only a single bubble in each cut, there is no possibility of cancellation. It is not uncommon for cancellations to occur amongst the box functions appearing in a cut. The propagators of a single box vanish as $1/z^2$, however, as in many of the cases we consider, the leading terms cancel amongst the boxes leaving a $1/z^4$ behaviour as $z \longrightarrow \infty$.

explicit all- n representation of the tree amplitudes [55]. This can be used to show that [40–42],

$$\begin{aligned} (h_i, h_j) = (+, +), (-, -), (+, -) : M^{\text{tree}}|_{z \rightarrow \infty} &\sim \frac{1}{z^2}, \\ (h_i, h_j) = (-, +) : M^{\text{tree}}|_{z \rightarrow \infty} &\sim z^6. \end{aligned} \quad (4.13)$$

Slightly more surprisingly this behaviour extends to $NMHV$ amplitudes also - at least up to seven points where we have checked the result explicitly. It is tempting to conjecture that this is true for all graviton tree amplitudes. We only need the behaviour up to seven points to test for bubbles in the six and seven point amplitudes.

We first consider the MHV case for arbitrary n . The singlet cuts are of the form,

$$M^{\text{tree}}((l_1)^+, 1^-, 2^-, 3^+, \dots r^+, (l_2)^+) \times M^{\text{tree}}((l_1)^-, (r+1)^+, \dots n^+, (l_2)^-). \quad (4.14)$$

When we shift this we find that each tree shifts as $1/z^2$, so the product behaves as $1/z^4$ at large- z and we can deduce that the bubble integral function $I_2(K_{1\dots r})$ has vanishing coefficient. The non-singlet cut is more involved,

$$\begin{aligned} \sum_h M^{\text{tree}}((-l_1)^{-h}, 2^-, 3^+, \dots r^+, (l_2)^h) \times M^{\text{tree}}((-l_2)^{-h}, (r+1)^+, \dots n^+, 1^-, (l_1)^h) = \\ M^{\text{tree}}((-l_1)^-, 2^-, 3^+, \dots r^+, (l_2)^+) \times M^{\text{tree}}((-l_2)^-, (r+1)^+, \dots n^+, 1^-, (l_1)^+) \\ \times \sum_{h \in \mathcal{S}'} \left(\frac{\langle 2 l_1 \rangle \langle 1 l_2 \rangle}{\langle 2 l_2 \rangle \langle 1 l_1 \rangle} \right)^{2h-4} \quad (4.15) \\ = \rho \times M^{\text{tree}}((-l_1)^-, 2^-, 3^+, \dots r^+, (l_2)^+) \times M^{\text{tree}}((-l_2)^-, (r+1)^+, \dots n^+, 1^-, (l_1)^+), \end{aligned}$$

where,

$$\rho = \left(\frac{\langle 2 l_2 \rangle \langle 1 l_1 \rangle - \langle 2 l_1 \rangle \langle 1 l_2 \rangle}{\langle 2 l_1 \rangle \langle 1 l_2 \rangle} \right)^8 = \left(\frac{\langle l_1 l_2 \rangle \langle 1 2 \rangle}{\langle 2 l_1 \rangle \langle 1 l_2 \rangle} \right)^8. \quad (4.16)$$

Under the shift the amplitudes scale as,

$$\begin{aligned} M^{\text{tree}}((-l_1)^-, 2^-, 3^+, \dots r^+, (l_2)^+) &\sim z^6, \\ M^{\text{tree}}((-l_2)^-, (r+1)^+, \dots n^+, 1^-, (l_1)^+) &\sim 1/z^2, \end{aligned} \quad (4.17)$$

however the ρ -factor scales, noting that $\langle l_1 l_2 \rangle$ is unshifted, as,

$$\rho \sim \frac{1}{z^8}, \quad (4.18)$$

and we find the non-singlet cuts scale as $1/z^4$, exactly as in the singlet case. Within the sum over the multiplet (4.15) the product of tree amplitudes scales as z^4 for any given state and the simplification only arises when the sum over the entire $\mathcal{N} = 8$ multiplet is performed.

We will now discuss the possible cuts of the six and seven-point $NMHV$ amplitudes. For any singlet cut,

$$M^{\text{tree}}((-l_1)^-, \dots, (l_2)^-) \times M^{\text{tree}}((-l_2)^+, \dots, (l_1)^+), \quad (4.19)$$

the trees both vanish as $1/z^2$ under the shift (4.12) and we deduce that bubble integral functions are absent from these cuts. Thus bubbles corresponding to singlet cuts are absent up to seven-points.

The non-singlet cuts are more involved. For the six-point amplitude, $M(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$ there are two types of cut corresponding to the cuts C_{234} and C_{34} . For the C_{234} cut the amplitudes are a product of an MHV and a \overline{MHV} . Summing over the multiplet gives an overall ρ factor just as in the MHV case and we deduce the coefficient of this bubble function is absent. The C_{34} cut is given by,

$$\sum_{h \in \mathcal{S}'} M^{\text{tree}}((-l_1)^{-h}, 3^-, 4^+, (l_2)^h) \times M^{\text{tree}}((-l_2)^{-h}, 1^-, 2^-, 5^+, 6^+, (l_1)^h). \quad (4.20)$$

The amplitude involving a state of helicity h behaves as,

$$M^{\text{tree}}((-l_1)^{-h}, 1^-, 2^-, 5^+, 6^+, (l_2)^h) \sim z^{2h+2}, \quad (4.21)$$

which is a natural refinement of (4.13) and can be checked using the form of the amplitude in appendix C. Thus the product of the two tree amplitudes in the cut (which will have states of $\pm h$) will always behave as z^4 and the corresponding scattering amplitude will contain bubble functions (or boundary terms). By explicit computation it can be seen that after including all the states from the $\mathcal{N} = 8$ supergravity multiplet we have,

$$\sum_{\mathcal{N}=8 \text{ multiplet}} M^{\text{tree}}((-l_1)^{-h}, 3^-, 4^+, (l_2)^h) \times M^{\text{tree}}((-l_2)^{-h}, 1^-, 2^-, 5^+, 6^+, (l_1)^h) \Big|_{z \rightarrow \infty} \longrightarrow 0, \quad (4.22)$$

and the bubble functions drop out. This calculation shows how the sum over the multiplet leads to the absence of bubble functions in the six-point $NMHV$ amplitude even though they are present in the contribution from any single state in the multiplet. For the seven-point amplitude $M(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ there are three types of cut: C_{234} , C_{345} and C_{34} . Of these, the large z behaviour of C_{234} and C_{345} can be checked using the six-point amplitudes verifying the absence of these bubbles.

5. Consequences and Conclusions

In this paper we have given further evidence that the one-loop perturbative expansion for $\mathcal{N} = 8$ supergravity is much closer to that of $\mathcal{N} = 4$ super Yang Mills than expected from power counting arguments. We argue that the one-loop amplitudes

are composed entirely of box integral functions and contain “no-triangle” (or bubble or rational) integral functions. We have provided evidence rather than a proof for this “no-triangle hypothesis”, but the evidence amounts to a proof for the six-point amplitudes. The evidence for n -point amplitudes with $n \geq 7$ based on unitarity, factorisation and IR behaviour is, for us, compelling.

The cancellation we observe is *not* “diagram-by-diagram” - at least not in any computational framework we are aware of. Individual diagrams appear to have loop momentum polynomials of degree $2(r - 4)$ and simplification only occurs when the diagrams are summed. The simplification observed is quite dramatic: to yield only boxes the simplification would be equivalent to a cancellation between terms such that the leading $(r - 4)$ terms in the loop momentum polynomials cancel.

The “no-triangle hypothesis” applies strictly to one-loop amplitudes only. However we expect it to have consequences for higher loops. For $\mathcal{N} = 8$ supergravity in $D = 4$ the four point amplitude is expected to diverge at five loops [4]. This argument is based on power counting and the known symmetries of the theory [56]. Specifically, the argument attempts to estimate the power of the loop momentum integral of individual higher loop diagrams and finds that they generically have twice the power of the equivalent Yang-Mills diagram. Cancellations between diagrams analogous to those occurring at one-loop would lead to a softer UV behaviour than this prediction with the theory possibly even being finite.

Presumably there is a symmetry underlying this simplification. We are not aware of any potential candidates for this symmetry. Although examining on-shell amplitudes has many advantages, the nature of the underlying symmetry is obscure in the amplitudes. The symmetries implied by the twistor duality [36] are one potential source, although originally the duality seemed to involve super-conformal rather than Einstein gravity [57]. Recently twistor strings involving Einstein gravity have been constructed [58] and it would be interesting to explore these for potential symmetries. If $\mathcal{N} = 8$ were “weak-weak” dual to a UV finite string theory then obviously the finiteness of $\mathcal{N} = 8$ supergravity would follow.

Acknowledgments

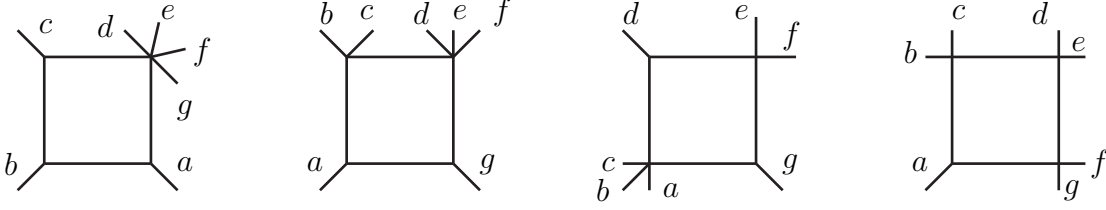
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A. Seven-Point Amplitude

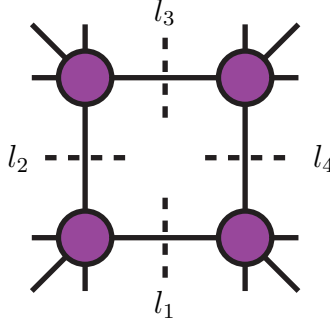
The seven-point $NMHV$ amplitude $M_7(1^-, 2^-, 3^-, 4^+, 5^+, 6^+, 7^+)$ can be expressed as a sum of scalar boxes together with rational coefficients;

$$M^{1\text{-loop}} = \sum_a c_a I_4^a. \quad (\text{A.1})$$

The scalar boxes can be of four types, three mass, two-mass-hard, two-mass-easy and one mass shown below with our choice of labelling.



The coefficients of the box-functions can be obtained by unitarity [33, 34]. Recently, it was observed that the box-coefficients can be efficiently obtained from the quadruple cut [35],



which yields the coefficient of the corresponding integral function,

$$c = \frac{1}{2} \sum_{h_i \in \mathcal{S}} \left(M^{\text{tree}}((-l_1)^{-h_1}, i_1, \dots, i_2, (\ell_2)^{h_2}) \times M^{\text{tree}}((-l_2)^{-h_2}, i_3, \dots, i_4, (\ell_3)^{h_3}) \right. \\ \left. \times M^{\text{tree}}((-l_3)^{-h_3}, i_5, \dots, i_6, (\ell_4)^{h_4}) \times M^{\text{tree}}((-l_4)^{-h_4}, i_7, \dots, i_8, (\ell_1)^{h_1}) \right). \quad (\text{A.2})$$

In this expression the sum is over all possible states of the $\mathcal{N} = 8$ multiplet and all possible helicity configurations for which the four tree amplitudes are non-zero. The four cut momenta are all on-shell, $\ell_i^2 = 0$. If the four tree amplitudes have four or more legs then this is solved for real momenta whereas if a corner has only three legs then the solution involves complex momenta. Alternately the box-coefficients of $\mathcal{N} = 8$ can be obtained from those of $\mathcal{N} = 4$ super Yang-Mills where for example, with the above labelling, the coefficients of the three-mass boxes are related by,

$$c_{N=8}^{[a,\{b,c\},\{d,e\},\{f,g\}]} = 2s_{bc}s_{de}s_{fg} \times c_{N=4}^{[a,\{b,c\},\{d,e\},\{f,g\}]} \times c_{N=4}^{[a,\{c,b\},\{e,d\},\{g,f\}]} \quad (\text{A.3})$$

To implement the quadruple cuts requires a knowledge of the tree amplitudes up to and including six-points, where two particles are states other than gravitons. The three, four and five points amplitudes are all MHV or \overline{MHV} amplitudes and relatively simple. For MHV amplitudes the amplitude with $n - 2$ gravitons and two non-graviton particles are related to the MHV amplitude by,

$$M(1_h^-, 2^-, 3^+, \dots, (n-1)^+, n_h^+) = \left(\frac{\langle 2n \rangle}{\langle 21 \rangle} \right)^{2h-4} M(1^-, 2^-, 3^+, \dots, (n-1)^+, n^+). \quad (\text{A.4})$$

For the six-point corners the tree may be MHV or $NMHV$. The six-graviton $NMHV$ tree amplitudes were computed recently [41, 17]. To complete the calculation of the box-coefficients we also need the six-point amplitudes with two non-gravitons. These are presented in appendix C.

A.1 Definitions

The coefficients of the boxes are expressed using spinor products. We use the notation $\langle j l \rangle \equiv \langle j^- | l^+ \rangle$, $[j l] \equiv \langle j^+ | l^- \rangle$, with $|i^\pm\rangle$ being massless Weyl spinors with momentum k_i and chirality \pm [59]. The spinor products are related to momentum invariants by $\langle i j \rangle [j i] = 2k_i \cdot k_j \equiv s_{ij}$. As in twistor-space studies we use the notation,

$$\lambda_i = |i^+\rangle, \quad \tilde{\lambda}_i = |i^-\rangle. \quad (\text{A.5})$$

We also define spinor strings,

$$\begin{aligned} [k|K_{i\dots j}|l\rangle &\equiv \langle k^+|K_{i\dots j}|l^+\rangle \equiv \langle l^-|K_{i\dots j}|k^-\rangle \equiv \langle l|K_{i\dots j}|k\rangle \equiv \sum_{a=i}^j [k a] \langle a l \rangle, \\ \langle k|K_{i\dots j}K_{m\dots n}|l\rangle &\equiv \langle k^-|K_{i\dots j}K_{m\dots n}|l^+\rangle = \sum_{a=i}^j \sum_{b=m}^n \langle k a \rangle [a b] \langle b l \rangle, \\ [k|K_{i\dots j}K_{m\dots n}|l] &\equiv \langle k^+|K_{i\dots j}K_{m\dots n}|l^-\rangle \equiv \sum_{a=i}^j \sum_{b=m}^n [k a] \langle a b \rangle [b l], \end{aligned} \quad (\text{A.6})$$

etc. We will often use the momentum invariants $s_{ij} = (k_i + k_j)^2$ and $t_{ijk} = (k_i + k_j + k_k)^2$.

A.2 Three Mass Boxes

The three mass boxes have one graviton attached to one corner and two gravitons to each of the others. The three-point corner is \overline{MHV} while the four-point corners are MHV . This means that all four corners are relatively simple and that different helicity configurations are also relatively simply related. In the case of L_6 there is a

summation over the full $\mathcal{N} = 8$ multiplet running in the loop. We get,

$$\begin{aligned}
c[a^+, \{b^+, c^+\}, \{d^-, e^+\}, \{f^-, g^-\}] &= L_0, \\
c[a^+, \{b^+, c^+\}, \{d^-, e^-\}, \{f^-, g^+\}] &= L_1 = \left(\frac{\langle f | K_{de} K_{bc} | a \rangle}{[e | K_{bc} | a] \langle f g \rangle} \right)^8 L_0, \\
c[a^+, \{b^+, c^-\}, \{d^+, e^+\}, \{f^-, g^-\}] &= L_2 = \left(\frac{\langle a c \rangle [d e]}{[e | K_{bc} | a]} \right)^8 L_0, \\
c[a^-, \{b^+, c^+\}, \{d^-, e^-\}, \{f^+, g^+\}] &= L_3 = \left(\frac{\langle a | K_{bc} K_{fg} | a \rangle}{[e | K_{bc} | a] \langle f g \rangle} \right)^8 L_0, \\
c[a^-, \{b^-, c^-\}, \{d^+, e^+\}, \{f^+, g^+\}] &= 0, \\
c[a^-, \{b^+, c^-\}, \{d^+, e^+\}, \{f^+, g^-\}] &= L_4 = \left(\frac{\langle a c \rangle \langle a g \rangle [d e]}{[e | K_{bc} | a] \langle f g \rangle} \right)^8 L_0, \\
c[a^-, \{b^-, c^+\}, \{d^-, e^+\}, \{f^+, g^+\}] &= L_5 = \left(\frac{[e | K_{fg} | a] \langle a b \rangle}{[e | K_{bc} | a] \langle f g \rangle} \right)^8 L_0, \\
c[a^+, \{b^-, c^+\}, \{d^-, e^+\}, \{f^-, g^+\}] &= L_6 \\
&= \left(\frac{\langle b a \rangle [e | K_{fg} | a] \langle f | K_{de} K_{bc} | a] - \langle f a \rangle [e | K_{bc} | a] \langle b | K_{de} K_{fg} | a]}{[e | K_{bc} | a] \langle f g \rangle \langle a | K_{bc} K_{de} | a]} \right)^8 L_0,
\end{aligned} \tag{A.7}$$

where,

$$L_0 = \frac{-s_{bc} s_{de} s_{fg} \langle g f \rangle^6 [e | K_{bc} | a]^8 (t_{abc} t_{fga} - s_{bc} s_{fg})^2}{2 [d e]^2 \langle b c \rangle^2 \prod_{x=b,c,g,f} \langle a x \rangle \prod_{y=d,e} [y | K_{fg} | a] [y | K_{bc} | a] \prod_{z=b,c} \langle z | K_{de} K_{fg} | a \rangle \prod_{w=f,g} \langle w | K_{de} K_{bc} | a \rangle}. \tag{A.8}$$

A.3 Two Mass Hard Boxes

The two mass hard boxes have two adjacent three-point corners, a four-point corner and a five-point corner. The four- and five-point corners are MHV and of the two three-point corners one is MHV and the other is \overline{MHV} . The two ways of assigning these give rise to the G_i and H_i terms below. Because all corners are either MHV or \overline{MHV} , the different helicity configurations are simply related. We get,

$$\begin{aligned}
c[a^-, \{b^-, c^-\}, \{d^+, e^+, f^+\}, g^+] &= G_0, \\
c[a^-, \{b^-, c^+\}, \{d^-, e^+, f^+\}, g^+] &= G_1 + H_1 = \left(\frac{[c|K_{abc}|d\rangle}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle a b \rangle [g|K_{abc}|d\rangle}{\langle b c \rangle t_{def}} \right)^8 H_0, \\
c[a^-, \{b^+, c^+\}, \{d^-, e^-, f^+\}, g^+] &= G_2 + H_2 = \left(\frac{[b c] \langle d e \rangle}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle d e \rangle [g|K_{abc}|d\rangle}{\langle b c \rangle t_{def}} \right)^8 H_0, \\
c[a^+, \{b^+, c^+\}, \{d^-, e^-, f^-\}, g^+] &= 0, \\
c[a^+, \{b^-, c^-\}, \{d^-, e^+, f^+\}, g^+] &= G_3 + H_3 = \left(\frac{[a|K_{abc}|d\rangle}{t_{abc}} \right)^8 G_0 + \left(\frac{[g|K_{abc}|d\rangle}{t_{def}} \right)^8 H_0, \\
c[a^+, \{b^-, c^+\}, \{d^-, e^-, f^+\}, g^+] &= G_4 + H_4 = \left(\frac{[a c] \langle d e \rangle}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle d e \rangle [g|K_{abc}|b\rangle}{\langle b c \rangle t_{def}} \right)^8 H_0, \\
c[a^+, \{b^-, c^-\}, \{d^+, e^+, f^+\}, g^-] &= G_5 + H_5 = \left(\frac{[a|K_{abc}|g\rangle}{t_{abc}} \right)^8 G_0 + H_0, \\
c[a^+, \{b^-, c^+\}, \{d^-, e^+, f^+\}, g^-] &= G_6 + H_6 = \left(\frac{\langle d g \rangle [a b]}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle d|K_{def}K_{abc}|b\rangle}{\langle b c \rangle t_{def}} \right)^8 H_0, \\
c[a^+, \{b^+, c^+\}, \{d^-, e^-, f^+\}, g^-] &= G_7 + H_7 = 0.G_0 + \left(\frac{t_{abc} \langle d e \rangle}{\langle b c \rangle t_{def}} \right)^8 H_0, \\
c[a^-, \{b^-, c^+\}, \{d^+, e^+, f^+\}, g^-] &= G_8 + H_8 = \left(\frac{[c|K_{abc}|g\rangle}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle a b \rangle}{\langle b c \rangle} \right)^8 H_0, \\
c[a^-, \{b^+, c^+\}, \{d^-, e^+, f^+\}, g^-] &= G_9 + H_9 = \left(\frac{\langle g d \rangle [b c]}{t_{abc}} \right)^8 G_0 + \left(\frac{\langle a|K_{bc}K_{def}|d\rangle}{\langle b c \rangle t_{def}} \right)^8 H_0,
\end{aligned} \tag{A.9}$$

where,

$$G_0 = \frac{s_{ag}^2 \langle b c \rangle t_{abc}^8 ([d e] \langle e f \rangle [f|K_{abc}|g] [a|K_{abc}|d] - \langle d e \rangle [e f] [d|K_{abc}|g] [a|K_{abc}|f])}{2 \bar{N}(a, b, c) N(d, e, f, g) [c|K_{abc}|g] [b|K_{abc}|g] [a|K_{abc}|d] [a|K_{abc}|e] [a|K_{abc}|f]}, \tag{A.10}$$

and,

$$H_0 = \frac{s_{ag}^2 s_{bc}^7 t_{def}^7 (\langle d e \rangle [e f] \langle f|K_{abc}|g] \langle a|K_{abc}|d] - [d e] \langle e f \rangle \langle d|K_{abc}|g] \langle a|K_{abc}|f])}{2 [b c]^2 N(d, e, f) \prod_{j=b,c} [j|K_{abc}|a] [j|K_{bc}K_{def}|g] \prod_{i=d,e,f} \langle i|K_{def}K_{bc}|a] [g|K_{abc}|i]}. \tag{A.11}$$

Here, $N(a, b, \dots m) = \prod_{i < j, i, j \in \{a, b, \dots m\}} \langle i j \rangle$ and $\bar{N}(a, b, \dots m) = \prod_{i < j, i, j \in \{a, b, \dots m\}} [i j]$

A.4 Two Mass Easy Boxes

The two mass easy boxes have two opposite \overline{MHV} three-point corners, an MHV four-point corner and a \overline{MHV} five-point corner. Again, the terms are relatively simple and related. They are,

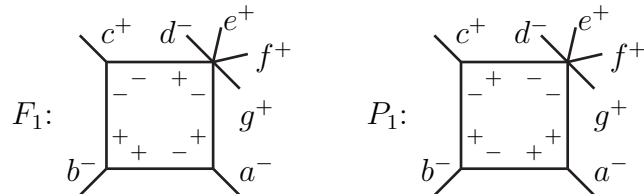
$$\begin{aligned}
c[\{a^-, b^-, c^-\}, d^+, \{e^+, f^+\}, g^+] &\equiv W_0, \\
c[\{a^-, b^-, c^+\}, d^+, \{e^+, f^-\}, g^+] &\equiv W_1 = \left(\frac{[c|K_{abc}|f\rangle}{t_{abc}} \right)^8 W_0, \\
c[\{a^-, b^-, c^+\}, d^+, \{e^+, f^+\}, g^-] &\equiv W_2 = \left(\frac{[c|K_{abc}|g\rangle}{t_{abc}} \right)^8 W_0, \\
c[\{a^-, b^+, c^+\}, d^+, \{e^-, f^-\}, g^+] &\equiv W_3 = \left(\frac{\langle e f \rangle [b c]}{t_{abc}} \right)^8 W_0, \\
c[\{a^-, b^+, c^+\}, d^+, \{e^-, f^+\}, g^-] &\equiv W_4 = \left(\frac{\langle e g \rangle [b c]}{t_{abc}} \right)^8 W_0, \\
c[\{a^-, b^+, c^+\}, d^-, \{e^+, f^+\}, g^-] &\equiv W_5 = \left(\frac{\langle d g \rangle [b c]}{t_{abc}} \right)^8 W_0, \\
c[\{a^+, b^+, c^+\}, d^-, \{e^-, f^-\}, g^+] &= 0, \\
c[\{a^+, b^+, c^+\}, d^-, \{e^-, f^+\}, g^-] &= 0,
\end{aligned} \tag{A.12}$$

where,

$$W_0 \equiv \frac{([g|K_{abc}|d\rangle][d|K_{abc}|g\rangle])^2 [ef]^2 (t_{abc})^7 (\langle g|K_{abc}k_a k_b k_c|d\rangle - \langle g|K_{abc}k_c k_b k_a|d\rangle)}{2[a b][a c][b c] \prod_{x=e,f} \langle x d \rangle \langle x g \rangle \prod_{x=a,b,c} [x|K_{abc}|g\rangle [x|K_{abc}|d\rangle] s_{ef}}. \tag{A.13}$$

A.5 One Mass Boxes

The one mass boxes have three three-point corners and one massive six-point corner. For each external helicity configuration there are (one or) two internal helicity configurations which cause the massive corner to be either MHV or $NMHV$. These give rise to the F_i and P_i terms, respectively. Taking the second external configuration below as an example, F_1 and P_1 come from,



The F_i terms have the same simplifications as noted above, while the calculational approach for the resulting P_i terms is discussed below.

$$\begin{aligned}
c[a^-, b^-, c^-, \{d^+, e^+, f^+, g^+\}] &= F_0 + 0, \\
c[a^-, b^-, c^+, \{d^-, e^+, f^+, g^+\}] &= F_1 + P_1 = \left(\frac{[c|K_{abc}|d\rangle}{t_{abc}} \right)^8 F_0 + P_1, \\
c[a^-, b^+, c^-, \{d^-, e^+, f^+, g^+\}] &= F_2 + P_2 = \left(\frac{[b|c]\langle d|b\rangle}{[a|c]\langle a|b\rangle} \right)^8 F_0 + P_2, \\
c[a^-, b^+, c^+, \{d^-, e^-, f^+, g^+\}] &= F_3 + P_3 = \left(\frac{\langle d|e\rangle[b|c]}{t_{abc}} \right)^8 F_0 + P_3, \\
c[a^+, b^-, c^+, \{d^-, e^-, f^+, g^+\}] &= F_4 + P_4 = \left(\frac{\langle d|e\rangle[a|c]}{t_{abc}} \right)^8 F_0 + P_4, \\
c[a^+, b^+, c^+, \{d^-, e^-, f^-, g^+\}] &= 0 + P_5,
\end{aligned} \tag{A.14}$$

where,

$$F_0 = \frac{t_{abc}^6 \langle a|b\rangle^2 \langle c|b\rangle^2 [a|g][d|e][f|K_{de}K_{abc}|a]}{4[a|c]\langle d|e\rangle\langle e|f\rangle\langle d|f\rangle\langle f|g\rangle[c|K_{abc}|g]\prod_{x=d,e,g}[a|K_{abc}|x]} + \text{Perm}(d, e, f, g). \tag{A.15}$$

For P_1 we use the form of the *NMHV* six-point tree amplitude in appendix C. We then get,

$$P_1^{[a,b,c,d,e,f,g]} = (\mathcal{T}_1^1) + (\mathcal{T}_1^2 + \mathcal{T}_1^3 + \mathcal{T}_1^4 + \mathcal{T}_1^5) |_{\{(efg)+(feg)+(gef)\}}, \tag{A.16}$$

with,

$$\begin{aligned}
\mathcal{T}_1^1 &= M_0[a, b, c, d, e, f, g] \\
&\equiv \frac{s_{ab}^2 \langle c|d\rangle \langle a|b\rangle^6 [b|c]^2 t_{efg}^7 \left([d|K_{efg}|e]\langle e|f\rangle\langle f|g\rangle[g|K_{abc}|c] - [d|K_{efg}|g]\langle g|f\rangle\langle f|e\rangle[e|K_{abc}|c] \right)}{2t_{abc}^2 \langle a|c\rangle [d|K_{abc}|a]\langle e|f\rangle\langle f|g\rangle\langle e|g\rangle \prod_{x=e,f,g}\langle c|K_{abc}K_{efg}|x\rangle[d|K_{efg}|x]}, \\
\mathcal{T}_1^2 &= \left(\frac{[e|K_{dfg}|d\rangle}{t_{dfg}} \right)^8 M_0[a, b, c, e, d, f, g], \\
\mathcal{T}_1^3 &= \frac{s_{ab}^2 \langle a|b\rangle^6 [b|c]^2 [f|g]^8 \langle c|K_{abc}|e\rangle \left(\langle e|K_{dfg}|d\rangle\langle d|f\rangle[f|g]\langle g|c\rangle - \langle e|K_{dfg}|g\rangle\langle g|f\rangle[f|d]\langle d|c\rangle \right)}{2t_{dfg} \langle a|c\rangle \langle a|e\rangle [d|f][f|g][d|g] \prod_{x=d,f,g}[x|K_{efg}|c][x|K_{efg}|e]}, \\
\mathcal{T}_1^4 &= \frac{-[e|K_{fg}|c]^7 [b|c]^2 \langle a|b\rangle^6 s_{de} s_{fg} s_{ab}^2}{2\langle c|K_{fg}K_{ade}|a\rangle \langle a|c\rangle [d|e]^2 \prod_{y=d,e}\langle c|K_{abc}|y\rangle\langle f|g\rangle^2 \langle g|c\rangle\langle f|c\rangle \prod_{x=f,g}\langle c|K_{abc}K_{de}|x\rangle[d|K_{fg}|c]}, \\
\mathcal{T}_1^5 &= \frac{\langle a|b\rangle^6 [b|c]^2 \langle c|d\rangle^7 [f|g]^6 s_{de} s_{fg} t_{abc} s_{ab}^2}{2\langle a|c\rangle \langle c|K_{abfg}K_{afg}|a\rangle \langle d|e\rangle^2 \langle c|e\rangle \prod_{y=f,g}[y|K_{abc}|c][y|K_{de}|c] \prod_{x=d,e}\langle x|K_{fg}K_{abc}|c\rangle}.
\end{aligned} \tag{A.17}$$

P_1 and P_2 are related by,

$$P_2^{[a,b,c,d,e,f,g]} = \left(\frac{\langle ca \rangle}{\langle bc \rangle} \right)^8 P_1^{[a,b,c,d,e,f,g]}. \quad (\text{A.18})$$

P_3 is obtained by using the $NMHV$ amplitude of Cachazo and Svrček [41]. We then get,

$$P_3^{[a,b,c,d,e,f,g]} = \sum_{i=1}^{13} \mathcal{T}_3^i, \quad (\text{A.19})$$

where,

$$\begin{aligned} \mathcal{T}_3^1 &= \frac{[ab]^2 [bc]^2 \langle de \rangle \langle a|K_{de}|f \rangle^7 (\langle a|K_{de}|f \rangle \langle g|K_{ef}|d \rangle \langle c|K_{ab}|g \rangle - \langle c|K_{ab}|d \rangle [fg] \langle ga \rangle t_{def})}{2 \langle ag \rangle [de] [ef]^2 \langle gc \rangle t_{def} \langle a|K_{ef}|d \rangle \prod_{x=c,g} \prod_{y=d,f} \langle x|K_{def}|y \rangle}, \\ \mathcal{T}_3^2 &= \frac{[ab]^2 [bc]^2 \langle ea \rangle (\langle da \rangle \langle c|K_{ab}|f \rangle + \langle ca \rangle \langle de \rangle [ef])^7}{2 \langle ca \rangle \langle dg \rangle \langle dc \rangle \langle c|K_{ab}|e \rangle [ef]^2 \langle gc \rangle \langle c|K_{dg}|f \rangle} \\ &\quad \times \frac{(\langle da \rangle \langle c|K_{ab}|f \rangle + \langle ca \rangle \langle de \rangle [ef]) \langle c|K_{ab}K_{ef}|g \rangle [gd] - \langle c|K_{ab}|d \rangle [fg] \langle gd \rangle \langle a|K_{ef}K_{dg}|c \rangle}{\langle a|K_{ef}K_{dg}|c \rangle \langle c|K_{ab}K_{ef}|d \rangle \prod_{x=c,g} \langle c|K_{ab}K_{ef}|x \rangle (\langle ga \rangle \langle c|K_{ab}|f \rangle + \langle ca \rangle \langle ge \rangle [ef])}, \\ \mathcal{T}_3^3 &= \frac{[ab]^2 [bc]^2 [gf] \langle e|K_{gf}K_{bc}|a \rangle^7}{2 \langle a|K_{bc}|d \rangle t_{abc} \langle fg \rangle \langle fe \rangle^2 \langle c|K_{ab}|d \rangle t_{gfe} \langle g|K_{fe}K_{bc}|a \rangle} \\ &\quad \times \frac{\langle e|K_{gf}K_{bc}|a \rangle \langle g|K_{fe}|d \rangle \langle dc \rangle + \langle cg \rangle \langle ed \rangle \langle a|K_{bc}|d \rangle t_{gfe}}{\prod_{x=e,g} \langle x|K_{efg}|d \rangle \langle c|K_{ab}K_{efg}|x \rangle}, \\ \mathcal{T}_3^4 &= \frac{[ab]^2 [bc]^2 \langle a|K_{bc}|f \rangle (\langle ce \rangle \langle a|K_{bc}|g \rangle + \langle ca \rangle \langle ef \rangle [fg])^7}{2 \langle ca \rangle [gd] \langle cf \rangle \langle fe \rangle^2 \prod_{x=d,g} \langle c|K_{ab}|x \rangle \langle c|K_{fe}|x \rangle} \\ &\quad \times \frac{(\langle ce \rangle \langle a|K_{bc}|g \rangle + \langle ca \rangle \langle ef \rangle [fg]) \langle c|K_{fe}|d \rangle \langle dg \rangle + \langle gc \rangle \langle ed \rangle [dg] \langle a|K_{dg}K_{ef}|c \rangle}{\langle a|K_{dg}K_{ef}|c \rangle \langle c|K_{ab}K_{fe}|c \rangle \langle c|K_{ab}K_{dg}|e \rangle (\langle ce \rangle \langle a|K_{bc}|d \rangle + \langle ca \rangle \langle ef \rangle [fd])}, \\ \mathcal{T}_3^5 &= \frac{[ab]^2 [bc]^2 \langle ae \rangle^7 \langle dg \rangle [fg]^7 t_{abc}}{2 [df] [dg] \langle ec \rangle t_{dfg} \prod_{x=a,c} \langle x|K_{dg}|f \rangle \prod_{y=d,g} \langle e|K_{dfg}|y \rangle}, \\ \mathcal{T}_3^6 &= \frac{[ab]^2 [bc]^2 \langle ca \rangle^7 \langle de \rangle^7 \langle ga \rangle [fg]^7 \langle a|K_{bc}|d \rangle}{2 \langle dc \rangle \prod_{x=f,g} \langle c|K_{ab}|x \rangle \langle ec \rangle \langle a|K_{fg}K_{de}|c \rangle \langle c|K_{de}|f \rangle \langle c|K_{ab}K_{fg}|e \rangle} \\ &\quad \times \frac{1}{(\langle da \rangle \langle c|K_{ab}|f \rangle + \langle ca \rangle \langle dg \rangle [gf]) (\langle ce \rangle \langle a|K_{bc}|g \rangle + \langle ca \rangle \langle ed \rangle [dg])}, \\ \mathcal{T}_3^7 &= \frac{[ab]^2 [bc]^2 \langle ae \rangle^7 \langle cd \rangle \langle a|K_{bc}|f \rangle^7 \langle c|K_{ab}|g \rangle}{2 \langle ca \rangle \langle ag \rangle [df] \langle a|K_{bc}|d \rangle \langle eg \rangle \langle a|K_{eg}K_{df}|c \rangle \langle a|K_{eg}|f \rangle \langle e|K_{df}K_{bc}|a \rangle} \\ &\quad \times \frac{1}{(\langle cg \rangle \langle a|K_{bc}|f \rangle + \langle ca \rangle \langle gd \rangle [df]) (\langle ea \rangle \langle c|K_{ab}|d \rangle + \langle ca \rangle \langle eg \rangle [gd])}, \\ \mathcal{T}_3^8 &= \frac{[ab]^2 [bc]^2 \langle de \rangle^7 \langle a|K_{bc}|f \rangle^7 [dg]}{2 \langle dg \rangle \langle c|K_{ab}|f \rangle t_{abc} \langle ge \rangle t_{deg} \prod_{x=d,g} \langle x|K_{deg}|f \rangle \prod_{y=a,c} \langle y|K_{abc}K_{dg}|e \rangle}, \end{aligned}$$

$$\mathcal{T}_3^9 = \frac{[ab]^2[bc]^2\langle ae\rangle^8\langle c|K_{ab}|f\rangle\langle a|K_{bc}|g\rangle^7}{2\langle ca\rangle\langle af\rangle[dg]\langle a|K_{bc}|d\rangle\langle ef\rangle^2\langle a|K_{ef}K_{dg}|c\rangle\langle a|K_{ef}K_{bc}|a\rangle\langle e|K_{dg}K_{bc}|a\rangle} \\ \times \frac{\langle de\rangle\langle gc\rangle\langle a|K_{bc}|d\rangle\langle a|K_{ef}|g\rangle - \langle eg\rangle\langle a|K_{bc}|g\rangle\langle cd\rangle\langle a|K_{ef}|d\rangle}{\prod_{x=d,g}\langle a|K_{ef}|x\rangle(\langle ea\rangle\langle c|K_{ab}|x\rangle + \langle ca\rangle\langle ef\rangle[fx])},$$

$$\mathcal{T}_3^{10} = \frac{[ab]^2[bc]^2\langle de\rangle^8[df]\langle a|K_{bc}|g\rangle^7\langle ae\rangle\langle cg\rangle t_{abc}\langle d|K_{ef}|g\rangle + \langle eg\rangle\langle a|K_{bc}|g\rangle\langle c|K_{ab}K_{ef}|d\rangle}{2\langle df\rangle\langle c|K_{ab}|g\rangle t_{abc}\langle ef\rangle^2 t_{def} \prod_{x=d,e} \prod_{y=a,c} \langle x|K_{def}|g\rangle\langle y|K_{abc}K_{def}|x\rangle},$$

$$\mathcal{T}_3^{11} = \frac{[ab]^2[bc]^2\langle a|K_{bc}|f\rangle^8\langle ce\rangle\langle da\rangle^7}{2\langle ca\rangle\langle a|K_{bc}|e\rangle\langle gd\rangle\langle ga\rangle[fe]^2\langle a|K_{dg}K_{ef}|c\rangle\langle a|K_{fe}K_{bc}|a\rangle\langle a|K_{dg}|f\rangle} \\ \times \frac{[fg]\langle c|K_{ab}|d\rangle\langle ag\rangle\langle d|K_{fe}K_{bc}|a\rangle + [fd]\langle da\rangle\langle c|K_{ab}|g\rangle\langle g|K_{fe}K_{bc}|a\rangle}{\prod_{x=d,g}\langle x|K_{fe}K_{bc}|a\rangle(\langle cx\rangle\langle a|K_{bc}|f\rangle + \langle ca\rangle\langle xe\rangle[ef])},$$

$$\mathcal{T}_3^{12} = \frac{[ab]^2[bc]^2[gf]^8\langle ge\rangle\langle da\rangle^7(-\langle a|K_{bc}|f\rangle\langle c|K_{ab}|d\rangle\langle d|K_{fe}|g\rangle + [fd]\langle da\rangle t_{abc}\langle c|K_{fe}|g\rangle)}{2[ge]\langle cd\rangle[fe]^2 t_{gfe} \prod_{x=a,c,d} \prod_{y=f,g} \langle x|K_{efg}|y\rangle},$$

$$\mathcal{T}_3^{13} = \frac{[ab]^2[bc]^2\langle ad\rangle\langle a|K_{bc}|g\rangle(\langle ea\rangle\langle c|K_{ab}|f\rangle + \langle ca\rangle\langle ed\rangle[df])^8}{2\langle ca\rangle \prod_{x=d,f} \langle c|K_{ab}|x\rangle[df]\langle eg\rangle\langle ce\rangle\langle gc\rangle\langle a|K_{df}K_{eg}|c\rangle\langle c|K_{eg}|f\rangle\langle c|K_{ab}K_{df}|e\rangle} \\ \times \frac{1}{(\langle ga\rangle\langle c|K_{ab}|f\rangle + \langle ca\rangle\langle gd\rangle[df])(\langle ce\rangle\langle a|K_{bc}|d\rangle + \langle ca\rangle\langle eg\rangle[gd])}. \quad (\text{A.20})$$

P_4 has the additional complication that we must sum over the full $\mathcal{N} = 8$ multiplet running in the loop. We obtain a form based on P_3 with relative factors for each \mathcal{T}_3^i . We also obtain one extra term which is not present in P_3 . We get,

$$P_4^{[a,b,c,d,e,f,g]} = \sum_{i_1}^{13} (Y_4^{i_1})^8 \mathcal{T}_3^{i_1} + \mathcal{T}_4^{14}, \quad (\text{A.21})$$

where,

$$\begin{aligned}
Y_4^1 &= -\frac{\langle b|K_{de}|f\rangle}{\langle a|K_{de}|f\rangle}, \\
Y_4^2 &= \frac{\langle bd\rangle\langle c|K_{ab}|f\rangle + \langle bc\rangle\langle de\rangle[ef]}{\langle da\rangle\langle c|K_{ab}|f\rangle + \langle ca\rangle\langle de\rangle[ef]}, \\
Y_4^3 &= -\frac{\langle b|K_{ac}K_{fg}|e\rangle}{\langle a|K_{bc}K_{fg}|e\rangle}, \\
Y_4^4 &= -\frac{\langle ce\rangle\langle b|K_{ac}|g\rangle + \langle cb\rangle\langle ef\rangle[fg]}{\langle ce\rangle\langle a|K_{bc}|g\rangle + \langle ca\rangle\langle ef\rangle[fg]}, \\
Y_4^5 &= \frac{\langle eb\rangle}{\langle ae\rangle}, \\
Y_4^6 &= \frac{\langle bc\rangle}{\langle ca\rangle}, \\
Y_4^7 &= \frac{\langle eb\rangle\langle a|K_{bc}|f\rangle - \langle ba\rangle\langle ed\rangle[df]}{\langle ae\rangle\langle a|K_{bc}|f\rangle}, \\
Y_4^8 &= -\frac{\langle b|K_{ac}|f\rangle}{\langle a|K_{bc}|f\rangle}, \\
Y_4^9 &= \frac{\langle be\rangle\langle a|K_{bc}|g\rangle + \langle ba\rangle\langle ed\rangle[dg]}{\langle ea\rangle\langle a|K_{bc}|g\rangle}, \\
Y_4^{10} &= -\frac{\langle b|K_{ac}|g\rangle}{\langle a|K_{bc}|g\rangle}, \\
Y_4^{11} &= \frac{\langle bd\rangle\langle a|K_{bc}|f\rangle + \langle ba\rangle\langle de\rangle[ef]}{\langle da\rangle\langle a|K_{bc}|f\rangle}, \\
Y_4^{12} &= \frac{\langle db\rangle}{\langle ad\rangle}, \\
Y_4^{13} &= \frac{\langle be\rangle\langle c|K_{ab}|f\rangle + \langle bc\rangle\langle ed\rangle[df]}{\langle ea\rangle\langle c|K_{ab}|f\rangle + \langle ca\rangle\langle ed\rangle[df]}, \\
\mathcal{T}_4^{14} &= \left(\frac{\langle ab\rangle}{\langle ca\rangle}\right)^8 \mathcal{T}_3^6(a \leftrightarrow c).
\end{aligned} \tag{A.22}$$

Last comes P_5 which has been obtained from the amplitude of Cachazo and Svrček by letting 5 and 6 be the internal gravitons. We get,

$$\begin{aligned}
P_5^{[a,b,c,d,e,f,g]} &= \mathcal{T}_5^1 + \mathcal{T}_5^1(d \leftrightarrow e) + \mathcal{T}_5^2 + \mathcal{T}_5^2(a \leftrightarrow c) \\
&+ \mathcal{T}_5^3 + \mathcal{T}_5^3(d \leftrightarrow e) + \mathcal{T}_5^3(a \leftrightarrow c) + \mathcal{T}_3^c(a \leftrightarrow c, d \leftrightarrow e) \\
&+ \mathcal{T}_5^4 + \mathcal{T}_5^4(d \leftrightarrow e) + \mathcal{T}_5^5 + \mathcal{T}_5^5(a \leftrightarrow c) + \mathcal{T}_5^6,
\end{aligned} \tag{A.23}$$

where,

$$\begin{aligned}
\mathcal{T}_5^1 &= \frac{[ab]^2[bc]^2\langle ef\rangle\langle d|K_{ef}|g\rangle^7(\langle d|K_{ef}|g\rangle\langle a|K_{fg}|e\rangle\langle c|K_{ab}|d\rangle - [de]\langle c|K_{ab}|g\rangle\langle ad\rangle t_{efg})}{2\langle da\rangle\langle cd\rangle[ef][fg]^2 t_{efg}\langle d|K_{fg}|e\rangle \prod_{x=e,g}\langle a|K_{efg}|x\rangle\langle c|K_{efg}|x\rangle}, \\
\mathcal{T}_5^2 &= \frac{[ab]^2[bc]^2\langle c|K_{ab}|g\rangle\langle f|K_{de}K_{bc}|a\rangle^7}{2\langle ca\rangle\langle ag\rangle\langle gf\rangle^2[ed]\langle a|K_{gf}K_{de}|c\rangle\langle a|K_{fg}K_{bc}|a\rangle} \\
&\quad \times \frac{\langle f|K_{de}K_{bc}|a\rangle\langle a|K_{gf}|e\rangle\langle ec\rangle - \langle fe\rangle\langle a|K_{bc}|e\rangle\langle a|K_{gf}K_{de}|c\rangle}{\prod_{x=d,e}\langle a|K_{bc}|x\rangle\langle a|K_{fg}|x\rangle(\langle fa\rangle\langle c|K_{ab}|x\rangle + \langle ca\rangle\langle fg\rangle[gx])}, \\
\mathcal{T}_5^3 &= \frac{[ab]^2[bc]^2\langle df\rangle^7\langle ea\rangle\langle c|K_{ab}|g\rangle^7\langle a|K_{bc}|d\rangle}{2\langle ca\rangle\langle dc\rangle[eg]\langle c|K_{ab}|e\rangle\langle fc\rangle\langle a|K_{eg}K_{df}|c\rangle\langle c|K_{df}|g\rangle\langle f|K_{eg}K_{ab}|c\rangle} \\
&\quad \times \frac{1}{(\langle da\rangle\langle c|K_{ab}|g\rangle + \langle ca\rangle\langle de\rangle[eg])(\langle cf\rangle\langle a|K_{bc}|e\rangle + \langle ca\rangle\langle fd\rangle[de])}, \\
\mathcal{T}_5^4 &= \frac{[ab]^2[bc]^2\langle df\rangle^8[dg]t_{abc}^7(\langle ef\rangle\langle a|K_{bc}|e\rangle\langle c|K_{ab}K_{fg}|d\rangle + \langle fa\rangle t_{abc}\langle ce\rangle\langle d|K_{fg}|e\rangle)}{2\langle gd\rangle\langle fg\rangle^2 t_{dfg} \prod_{x=d,f} \prod_{y=a,c} \langle x|K_{dfg}|e\rangle\langle y|K_{abc}|e\rangle\langle y|K_{abc}K_{dfg}|x\rangle}, \\
\mathcal{T}_5^5 &= \frac{[ab]^2[bc]^2\langle a|K_{bc}|g\rangle^8\langle cf\rangle\langle ed\rangle^7}{2\langle ca\rangle\langle a|K_{bc}|f\rangle\langle ae\rangle\langle ad\rangle[gf]^2\langle a|K_{de}K_{gf}|c\rangle\langle a|K_{de}|g\rangle} \\
&\quad \times \frac{\langle c|K_{ab}|g\rangle[ed]\langle da\rangle\langle e|K_{fg}K_{bc}|a\rangle + [ge]\langle de\rangle\langle c|K_{ab}|d\rangle\langle a|K_{fg}K_{bc}|a\rangle}{\langle a|K_{fg}K_{bc}|a\rangle \prod_{x=d,e}(\langle cx\rangle\langle a|K_{bc}|g\rangle + \langle ca\rangle\langle xf\rangle[fg])\langle x|K_{fg}K_{bc}|a\rangle}, \\
\mathcal{T}_5^6 &= \frac{[ab]^2[bc]^2\langle de\rangle t_{abc}\langle f|K_{de}|g\rangle^8}{2[de][dg][eg]\langle fa\rangle\langle fc\rangle t_{deg}\langle a|K_{de}|g\rangle\langle c|K_{de}|g\rangle\langle f|K_{eg}|d\rangle\langle f|K_{dg}|e\rangle}.
\end{aligned} \tag{A.24}$$

B. Relations Between Box Coefficients

The box-coefficients exhibit a large number of relations. As a consequence of the IR structure many combinations can be used to create expressions for the tree amplitudes. This has in fact been used to obtain relatively compact formulae for tree amplitudes [19, 60] and gave rise to the BCFW recursion relations [39]. Since the IR relations are satisfied, the box-coefficients are related to the tree amplitudes and in fact yield a form of the tree amplitude which is equivalent to that obtained via recursion. Before commencing it is convenient to define scaled box-coefficients⁵,

$$\begin{aligned}
\hat{c}^{1m}[a, b, c, \{d, e, f, g\}] &\equiv \frac{c^{1m}[a, b, c, \{d, e, f, g\}]}{s_{ab}s_{bc}}, \\
\hat{c}^{2m\ h}[a, \{b, c\}, \{d, e, f\}, g] &\equiv \frac{c^{2m\ h}[a, \{b, c\}, \{d, e, f\}, g]}{s_{ga}t_{abc}}, \\
\hat{c}^{2m\ e}[a, \{b, c\}, d, \{e, f, g\}] &\equiv \frac{c^{2m\ e}[a, \{b, c\}, d, \{e, f, g\}]}{t_{abc}t_{bcd} - s_{bc}t_{efg}}.
\end{aligned} \tag{B.1}$$

⁵The scaling factors are essentially the momentum prefactors appearing in the integral functions.

We will also use this notation to indicate the scaled functions which define the box-coefficients.

B.1 Expressions for tree amplitudes

For the seven-point one-loop *NMHV* amplitude there are circa 1000 independent boxes with each box coefficient containing two or more terms. We can extract the tree by looking at the coefficient of a specific logarithm: there being three independent choices: $\ln(-s_{12})$, $\ln(-s_{45})$ and $\ln(-s_{34})$. If we take the coefficient of $\ln(-s_{12})$ then only a subset of boxes will contribute to this. Contained within this is a further subset where the legs 1 and 2 are massless and the boxes are the one-mass and two-mass hard.

$$\begin{aligned}
M_7^{\text{tree}} = & \left(\hat{F}_0^{[1,2,3,4,5,6,7]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{F}_1^{[1,2,4,3,5,6,7]} + \hat{F}_1^{[1,2,5,3,4,6,7]} + \hat{F}_1^{[1,2,6,3,5,4,7]} + \hat{F}_1^{[1,2,7,3,5,6,4]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{P}_1^{[1,2,3,4,5,6,7]} + \hat{P}_1^{[1,2,3,5,4,6,7]} + \hat{P}_1^{[1,2,3,6,5,4,7]} + \hat{P}_1^{[1,2,3,7,5,6,4]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{G}_8^{[2,3,4,5,6,7,1]} + \hat{G}_8^{[2,3,5,4,6,7,1]} + \hat{G}_8^{[2,3,6,5,4,7,1]} + \hat{G}_8^{[2,3,7,5,6,4,1]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{H}_8^{[2,3,4,5,6,7,1]} + \hat{H}_8^{[2,3,5,4,6,7,1]} + \hat{H}_8^{[2,3,6,5,4,7,1]} + \hat{H}_8^{[2,3,7,5,6,4,1]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{G}_9^{[2,4,5,3,6,7,1]} + \hat{G}_9^{[2,4,6,3,5,7,1]} + \hat{G}_9^{[2,4,7,3,6,5,1]} + \hat{G}_9^{[2,5,6,3,4,7,1]} \right. \\
& \quad \left. + \hat{G}_9^{[2,5,7,3,4,6,1]} + \hat{G}_9^{[2,6,7,3,4,5,1]} + \{1 \leftrightarrow 2\} \right) \\
& + \left(\hat{H}_9^{[2,4,5,3,6,7,1]} + \hat{H}_9^{[2,4,6,3,5,7,1]} + \hat{H}_9^{[2,4,7,3,6,5,1]} + \hat{H}_9^{[2,5,6,3,4,7,1]} \right. \\
& \quad \left. + \hat{H}_9^{[2,5,7,3,4,6,1]} + \hat{H}_9^{[2,6,7,3,4,5,1]} + \{1 \leftrightarrow 2\} \right).
\end{aligned} \tag{B.2}$$

Within this set there are two subsets which each yield the tree, e.g.

$$\begin{aligned}
\hat{F}_0^{[1,2,3,4,5,6,7]} + & \left(\sum_{(a,b,c,d) \in \mathcal{S}_1} \hat{F}_1^{[1,2,a,3,b,c,d]} \right) + \left(\sum_{(a,b,c,d) \in \mathcal{S}_1} \hat{G}_8^{[1,3,a,b,c,d,2]} \right) + \left(\sum_{(a,b,c,d) \in \mathcal{S}_2} \hat{G}_9^{[1,a,b,3,c,d,2]} \right) \\
& + \left(\sum_{(a,b,c,d) \in \mathcal{S}_1} \hat{H}_8^{[2,3,a,b,c,d,1]} \right) + \left(\sum_{(a,b,c,d) \in \mathcal{S}_2} \hat{H}_9^{[2,a,b,3,c,d,1]} \right) + \left(\sum_{(a,b,c,d) \in \mathcal{S}_1} \hat{P}_1^{[2,1,3,a,b,c,d]} \right),
\end{aligned} \tag{B.3}$$

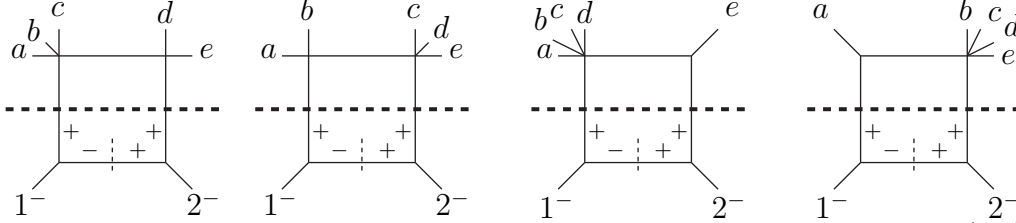
where,

$$\begin{aligned}
\mathcal{S}_1 = & \{(4, 5, 6, 7), (5, 4, 6, 7), (6, 4, 5, 7), (7, 4, 5, 6)\}, \\
\mathcal{S}_2 = & \{(4, 5, 6, 7), (4, 6, 5, 7), (4, 7, 5, 6), (5, 6, 4, 7), (5, 7, 4, 6), (6, 7, 4, 5)\}.
\end{aligned} \tag{B.4}$$

This provides a fairly compact realisation of the seven-point tree amplitude containing twenty-nine individual terms. This collection of terms corresponds exactly to the terms that would be obtained from a recursive calculation using legs 1 and 2 for the

recursion. The above expression has all the necessary symmetries although not all are manifest.

This subset of the box-coefficients corresponds to those terms where legs 1 and 2 are isolated at massless corners and where these corners have the helicity structure indicated.



Alternate expressions may be obtained by examining the coefficients of $\ln(s_{45})$ and $\ln(s_{34})$.

B.2 The coefficient of $\ln(-t_{123})$

A different type of relationship holds for the box-coefficients which contribute to the soft divergence $\ln(-t_{123})/\epsilon$. These soft divergences are absent so the box coefficients are conspiring to make them cancel. There are three types of box giving this divergence: two-mass easy, two-mass hard and one-mass. Specifically we must have,

$$\left(\sum_{Z(1,2,3)} \sum_{Z(4,5,6,7)} \hat{c}_{(1^-, \{2^-, 3^-\}, \{4^+, 5^+, 6^+\}, 7^+)}^{2mh} \right) - \left(\sum_{Z(1,2,3)} \hat{c}_{(1^-, 2^-, 3^-, \{4^+, 5^+, 6^+\}, 7^+)}^{1m} \right) - \left(\sum_{P(4,5,6,7)} \hat{c}_{(\{1^-, 2^-, 3^-\}, 4^+, \{5^+, 6^+\}, 7^+)}^{2me} \right) = 0, \quad (\text{B.5})$$

where Z denotes cyclic permutations and,

$$P_{(4,5,6,7)} = \{(4, 5, 6, 7), (4, 7, 5, 6), (4, 6, 7, 5), (5, 4, 6, 7), (5, 4, 7, 6), (6, 4, 5, 7)\}.$$

This relationship is indeed satisfied since,

$$\begin{aligned} \sum_{Z(1,2,3)} \sum_{Z(4,5,6,7)} \hat{c}_{(1^-, \{2^-, 3^-\}, \{4^+, 5^+, 6^+\}, 7^+)}^{2mh} &= 2 \sum_{Z(1,2,3)} \hat{c}_{(1^-, 2^-, 3^-, \{4^+, 5^+, 6^+\}, 7^+)}^{1m}, \\ \sum_{P(4,5,6,7)} \hat{c}_{(\{1^-, 2^-, 3^-\}, 4^+, \{5^+, 6^+\}, 7^+)}^{2me} &= \sum_{Z(1,2,3)} \hat{c}_{(1^-, 2^-, 3^-, \{4^+, 5^+, 6^+\}, 7^+)}^{1m}, \end{aligned} \quad (\text{B.6})$$

although clearly these two constraints are considerably stronger than the single constraint (B.5).

C. Six-Point Tree amplitudes involving non-gravitons

To calculate the cuts of the seven-point amplitude we need the six-point $NMHV$ amplitudes where one pair of particles is of arbitrary type. The six-point amplitude

is given in the form,

$$M(1^-, 2^-, (l_1)_h^-, (l_2)_h^+, 5^+, 6^+) = \sum_{i=1}^{14} T_i(h) = \sum_{i=1}^{14} A_i(X_i)^{2h}, \quad (\text{C.1})$$

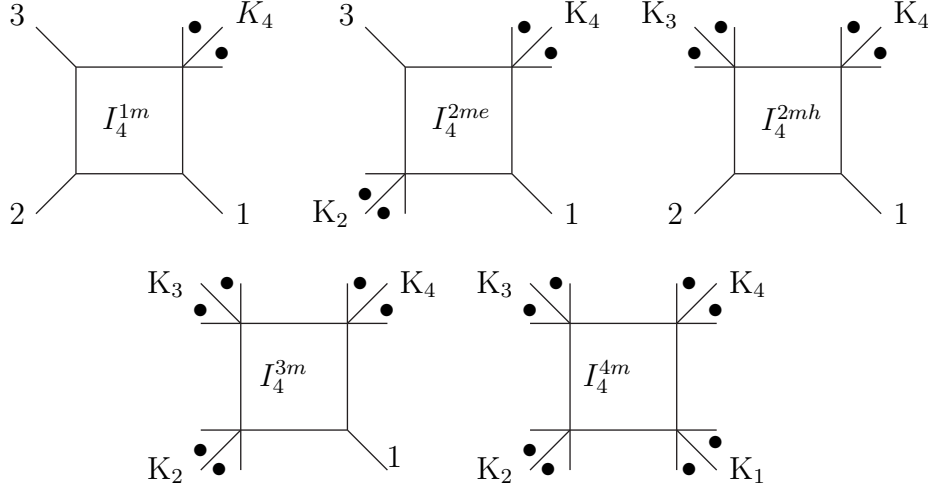
where $h = 2$ for a graviton, $h = 3/2$ for a gravitino, $h = 1$ for a vector, $h = 1/2$ for a Dirac fermion and $h = 0$ for a scalar particle. The expression is also valid for negative values of h provided we recognise that this corresponds to a particle of the opposite helicity e.g. $1_-^2 \equiv 1_{+2}^+$. The explicit forms of the T_i are given by,

$$\begin{aligned} T_1 &= \frac{-i \langle 12 \rangle^7 \langle 5l_2 \rangle [2l_1] [56]^7}{\langle 1l_1 \rangle \langle 2l_1 \rangle \langle 1|P_{12l_1}|5 \rangle \langle 1|P_{2l_1}|l_2 \rangle \langle 2|P_{2l_1}|6 \rangle \langle l_1|P_{12l_1}|6 \rangle [5l_2] [6l_2] t_{12l_1}} \left[\delta_{h,2} \right], \\ T_2 &= \frac{-i \langle 2l_1 \rangle \langle 1|P_{26l_1}|6 \rangle^8 [5l_2]}{\langle 15 \rangle \langle 1l_2 \rangle \langle 5l_2 \rangle \langle 1|P_{26l_1}|2 \rangle \langle 1|P_{26l_1}|l_1 \rangle \langle 5|P_{26l_1}|6 \rangle \langle l_2|P_{26l_1}|6 \rangle [26] [2l_1] [6l_1] t_{26l_1}} \left[\frac{-i \langle 1l_2 \rangle [6l_1]}{\langle 1|P_{26l_1}|6 \rangle} \right]^A, \\ T_3 &= \frac{-i \langle 12 \rangle^7 \langle 5l_1 \rangle [2l_2] [56]^7}{\langle 1l_2 \rangle \langle 2l_2 \rangle \langle 1|P_{56l_1}|5 \rangle \langle 1|P_{56l_1}|l_1 \rangle \langle 2|P_{56l_1}|6 \rangle \langle l_2|P_{56l_1}|6 \rangle [5l_1] [6l_1] t_{56l_1}} \left[\delta_{h,-2} \right], \\ T_4 &= \frac{i \langle 1l_1 \rangle^7 \langle 25 \rangle [56]^7 [l_1 l_2]}{\langle 1l_2 \rangle \langle l_1 l_2 \rangle \langle 1|P_{256}|2 \rangle \langle 1|P_{256}|5 \rangle \langle l_1|P_{256}|6 \rangle \langle l_2|P_{256}|6 \rangle [25] [26] t_{256}} \left[\frac{i \langle 1l_2 \rangle}{\langle 1l_1 \rangle} \right]^A, \\ T_5 &= \frac{i \langle 12 \rangle^7 \langle l_1 l_2 \rangle [25] [6l_2]^7}{\langle 15 \rangle \langle 25 \rangle \langle 1|P_{125}|l_1 \rangle \langle 1|P_{125}|l_2 \rangle \langle 2|P_{125}|6 \rangle \langle 5|P_{125}|6 \rangle [6l_1] [l_1 l_2] t_{125}} \left[\frac{i [6l_1]}{[6l_2]} \right]^A, \\ T_6 &= \frac{-i \langle 1l_1 \rangle^7 \langle 2l_2 \rangle [5l_1] [6l_2]^7}{\langle 15 \rangle \langle 5l_1 \rangle \langle 1|P_{15l_1}|2 \rangle \langle 1|P_{15l_1}|l_2 \rangle \langle 5|P_{15l_1}|6 \rangle \langle l_1|P_{15l_1}|6 \rangle [26] [2l_2] t_{15l_1}} \left[\frac{-i \langle 1|P_{15l_1}|6 \rangle}{\langle 1l_1 \rangle [6l_2]} \right]^A, \\ T_7 &= \frac{i \langle 1|P_{156}|l_2 \rangle^7 (\langle 1l_2 \rangle \langle 2l_1 \rangle \langle 5|P_{156}|l_1 \rangle [2l_2] - \langle 1l_1 \rangle \langle 2l_2 \rangle \langle 5|P_{156}|l_2 \rangle [2l_1]) [56]}{\langle 16 \rangle^2 \langle 56 \rangle \langle 1|P_{156}|2 \rangle \langle 1|P_{156}|l_1 \rangle \langle 5|P_{156}|2 \rangle \langle 5|P_{156}|l_1 \rangle \langle 5|P_{156}|l_2 \rangle [2l_1] [2l_2] [l_1 l_2] t_{156}} \left[\frac{i \langle 1|P_{156}|l_1 \rangle}{\langle 1|P_{156}|l_2 \rangle} \right]^A, \\ T_8 &= \frac{i \langle 1|P_{25l_1}|5 \rangle^7 (\langle 15 \rangle \langle 2l_1 \rangle \langle l_2|P_{25l_1}|l_1 \rangle [25] - \langle 1l_1 \rangle \langle 25 \rangle \langle l_2|P_{25l_1}|5 \rangle [2l_1]) [6l_2]}{\langle 16 \rangle^2 \langle 6l_2 \rangle \langle 1|P_{25l_1}|2 \rangle \langle 1|P_{25l_1}|l_1 \rangle \langle l_2|P_{25l_1}|2 \rangle \langle l_2|P_{25l_1}|5 \rangle \langle l_2|P_{25l_1}|l_1 \rangle [25] [2l_1] [5l_1] t_{25l_1}} \left[\frac{i \langle 1l_2 \rangle [5l_1]}{\langle 1|P_{25l_1}|5 \rangle} \right]^A, \\ T_9 &= \frac{i \langle 1l_1 \rangle^8 [5l_2]^7 (\langle 1l_2 \rangle \langle 25 \rangle \langle l_1|P_{6l_1}|2 \rangle [5l_2] - \langle 12 \rangle \langle 5l_2 \rangle \langle l_1|P_{6l_1}|l_2 \rangle [25]) [6l_1]}{\langle 16 \rangle^2 \langle 6l_1 \rangle \langle 1|P_{6l_1}|2 \rangle \langle 1|P_{6l_1}|5 \rangle \langle 1|P_{6l_1}|l_2 \rangle \langle l_1|P_{6l_1}|2 \rangle \langle l_1|P_{6l_1}|5 \rangle \langle l_1|P_{6l_1}|l_2 \rangle [25] [2l_2] t_{16l_1}} \left[\frac{i \langle 1|P_{6l_1}|5 \rangle}{\langle 1l_1 \rangle [5l_2]} \right]^A, \\ T_{10} &= \frac{i \langle 12 \rangle^8 [26] [5l_2]^7 (\langle 1l_2 \rangle \langle 5l_1 \rangle \langle 2|P_{126}|l_1 \rangle [5l_2] - \langle 1l_1 \rangle \langle 5l_2 \rangle \langle 2|P_{126}|l_2 \rangle [5l_1])}{\langle 16 \rangle^2 \langle 26 \rangle \langle 1|P_{126}|5 \rangle \langle 1|P_{126}|l_1 \rangle \langle 1|P_{126}|l_2 \rangle \langle 2|P_{126}|5 \rangle \langle 2|P_{126}|l_1 \rangle \langle 2|P_{126}|l_2 \rangle [5l_1] [l_1 l_2] t_{126}} \left[\frac{i [5l_1]}{[5l_2]} \right]^A, \\ T_{11} &= \frac{i \langle 15 \rangle \langle 2l_1 \rangle^7 [56]^8 (\langle 2l_2 \rangle \langle l_1|P_{156}|5 \rangle [2l_1] [6l_2] - \langle 2l_1 \rangle \langle l_2|P_{156}|5 \rangle [2l_2] [6l_1])}{\langle 2l_2 \rangle \langle l_1 l_2 \rangle \langle 2|P_{156}|5 \rangle \langle 2|P_{156}|6 \rangle \langle l_1|P_{156}|5 \rangle \langle l_1|P_{156}|6 \rangle \langle l_2|P_{156}|5 \rangle \langle l_2|P_{156}|6 \rangle [15] [16]^2 t_{156}} \left[\frac{i \langle 2l_2 \rangle}{\langle 2l_1 \rangle} \right]^A, \\ T_{12} &= \frac{-i \langle 1l_2 \rangle \langle 2l_1 \rangle^7 (\langle 25 \rangle \langle l_1|P_{25l_1}|l_2 \rangle [2l_1] [56] + \langle 2l_1 \rangle \langle 5|P_{25l_1}|l_2 \rangle [25] [6l_1]) [6l_2]^8}{\langle 25 \rangle \langle 5l_1 \rangle \langle 2|P_{25l_1}|6 \rangle \langle 2|P_{25l_1}|l_2 \rangle \langle 5|P_{25l_1}|6 \rangle \langle 5|P_{25l_1}|l_2 \rangle \langle l_1|P_{25l_1}|6 \rangle \langle l_1|P_{25l_1}|l_2 \rangle [1l_2] [16]^2 t_{25l_1}} \left[\frac{i \langle 2|P_{25l_1}|6 \rangle}{\langle 1l_2 \rangle [6l_2]} \right]^A, \\ T_{13} &= \frac{-i \langle 1l_1 \rangle \langle 2|P_{16l_1}|6 \rangle^7 (\langle 25 \rangle \langle l_2|P_{16l_1}|l_1 \rangle [26] [5l_2] + \langle 5l_2 \rangle \langle 2|P_{16l_1}|l_1 \rangle [25] [6l_2])}{\langle 25 \rangle \langle 2l_2 \rangle \langle 5l_2 \rangle \langle 2|P_{16l_1}|l_1 \rangle \langle 5|P_{16l_1}|6 \rangle \langle 5|P_{16l_1}|l_1 \rangle \langle l_2|P_{16l_1}|6 \rangle \langle l_2|P_{16l_1}|l_1 \rangle [16]^2 [1l_1] t_{16l_1}} \left[\frac{i \langle l_2 \rangle [6l_1]}{\langle 2|P_{16l_1}|6 \rangle} \right]^A, \\ T_{14} &= \frac{i \langle 12 \rangle \langle l_1|P_{126}|6 \rangle^7 (\langle 5l_2 \rangle \langle l_1|P_{126}|2 \rangle [5l_1] [6l_2] - \langle 5l_1 \rangle \langle l_2|P_{126}|2 \rangle [5l_2] [6l_1])}{\langle 5l_1 \rangle \langle 5l_2 \rangle \langle l_1 l_2 \rangle \langle 5|P_{126}|2 \rangle \langle 5|P_{126}|6 \rangle \langle l_1|P_{126}|2 \rangle \langle l_2|P_{126}|2 \rangle \langle l_2|P_{126}|6 \rangle [12] [16]^2 t_{126}} \left[\frac{i \langle l_2|P_{126}|6 \rangle}{\langle l_1|P_{126}|6 \rangle} \right]^A, \end{aligned} \quad (\text{C.2})$$

where $A = 4 - 2h$.

D. Integral Functions

D.1 Box Functions



The scalar box integrals considered here have vanishing internal masses, but may have up to four non-vanishing external masses. Again by external masses we mean off-shell legs with $K^2 \neq 0$. These integrals are defined and given in [61] (the four-mass box was computed by Denner, Nierste, and Scharf [62]) and are shown in the figures above.

The scalar box integral is,

$$I_4 = -i (4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{p^2 (p - K_1)^2 (p - K_1 - K_2)^2 (p + K_4)^2} . \quad (\text{D.1})$$

The external momentum arguments, K_i , are sums of external momenta k_i . In general the integrals are functions of the momentum invariants K_i^2 together with $S \equiv (K_1 + K_2)^2$ and $T = (K_2 + K_3)^2$. The no-mass box is, to $\mathcal{O}(\epsilon^0)$,

$$I_4^{0m}[1] = r_\Gamma \frac{1}{st} \left\{ \frac{2}{\epsilon^2} \left[(-s)^{-\epsilon} + (-t)^{-\epsilon} \right] - \ln^2 \left(\frac{-s}{-t} \right) - \pi^2 \right\} , \quad (\text{D.2})$$

where $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ are the usual Mandelstam variables. The factor r_Γ arises within dimensional regularisation and is,

$$r_\Gamma = \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1 + \epsilon) \Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} . \quad (\text{D.3})$$

This function appears only in four-point amplitudes with massless particles.

With the labelling of legs shown above, the scalar box integrals, I_4 , expanded to $\mathcal{O}(\epsilon^0)$ for the different cases reduce to,

$$I_4^{1m} = \frac{-2r_\Gamma}{ST} \left\{ -\frac{1}{\epsilon^2} \left[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right] \right. \\ \left. + \text{Li}_2 \left(1 - \frac{K_4^2}{S} \right) + \text{Li}_2 \left(1 - \frac{K_4^2}{T} \right) + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) + \frac{\pi^2}{6} \right\}, \quad (\text{D.4})$$

$$I_4^{2me} = \frac{-2r_\Gamma}{ST - K_2^2 K_4^2} \left\{ -\frac{1}{\epsilon^2} \left[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right] \right. \\ + \text{Li}_2 \left(1 - \frac{K_2^2}{S} \right) + \text{Li}_2 \left(1 - \frac{K_2^2}{T} \right) + \text{Li}_2 \left(1 - \frac{K_4^2}{S} \right) \\ \left. + \text{Li}_2 \left(1 - \frac{K_4^2}{T} \right) - \text{Li}_2 \left(1 - \frac{K_2^2 K_4^2}{ST} \right) + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) \right\}, \quad (\text{D.5})$$

$$I_4^{2mh} = \frac{-2r_\Gamma}{ST} \left\{ -\frac{1}{\epsilon^2} \left[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right] \right. \\ - \frac{1}{2\epsilon^2} \frac{(-K_3^2)^{-\epsilon} (-K_4^2)^{-\epsilon}}{(-S)^{-\epsilon}} + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) + \text{Li}_2 \left(1 - \frac{K_3^2}{T} \right) + \text{Li}_2 \left(1 - \frac{K_4^2}{T} \right) \left. \right\}, \quad (\text{D.6})$$

$$I_4^{3m} = \frac{-2r_\Gamma}{ST - K_2^2 K_4^2} \left\{ -\frac{1}{\epsilon^2} \left[(-S)^{-\epsilon} + (-T)^{-\epsilon} - (-K_2^2)^{-\epsilon} - (-K_3^2)^{-\epsilon} - (-K_4^2)^{-\epsilon} \right] \right. \\ - \frac{1}{2\epsilon^2} \frac{(-K_2^2)^{-\epsilon} (-K_2^2)^{-\epsilon}}{(-T)^{-\epsilon}} - \frac{1}{2\epsilon^2} \frac{(-K_3^2)^{-\epsilon} (-K_4^2)^{-\epsilon}}{(-T)^{-\epsilon}} + \frac{1}{2} \ln^2 \left(\frac{S}{T} \right) \\ \left. + \text{Li}_2 \left(1 - \frac{K_2^2}{S} \right) + \text{Li}_2 \left(1 - \frac{K_4^2}{T} \right) - \text{Li}_2 \left(1 - \frac{K_2^2 K_4^2}{ST} \right) \right\}, \quad (\text{D.7})$$

$$I_4^{4m} = \frac{-r_\Gamma}{S T \rho} \left\{ -\text{Li}_2 \left(\frac{1}{2} (1 - \lambda_1 + \lambda_2 + \rho) \right) + \text{Li}_2 \left(\frac{1}{2} (1 - \lambda_1 + \lambda_2 - \rho) \right) \right. \\ - \text{Li}_2 \left(-\frac{1}{2\lambda_1} (1 - \lambda_1 - \lambda_2 - \rho) \right) + \text{Li}_2 \left(-\frac{1}{2\lambda_1} (1 - \lambda_1 - \lambda_2 + \rho) \right) \\ \left. - \frac{1}{2} \ln \left(\frac{\lambda_1}{\lambda_2^2} \right) \ln \left(\frac{1 + \lambda_1 - \lambda_2 + \rho}{1 + \lambda_1 - \lambda_2 - \rho} \right) \right\}, \quad (\text{D.8})$$

where,

$$\rho \equiv \sqrt{1 - 2\lambda_1 - 2\lambda_2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2}, \quad (\text{D.9})$$

and,

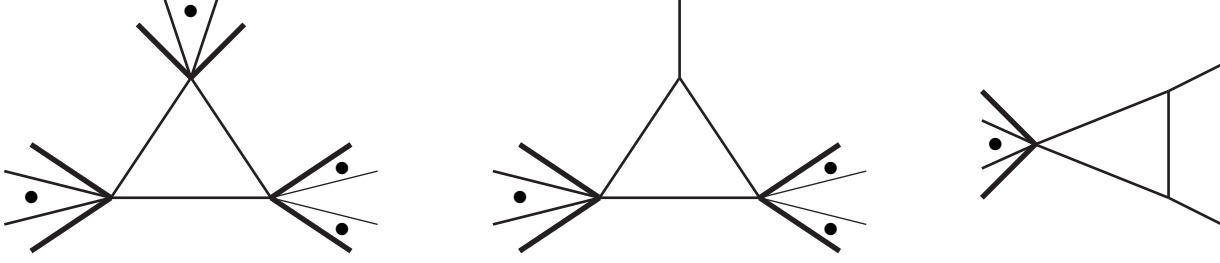
$$\lambda_1 = \frac{K_2^2 K_4^2}{S T}, \quad \lambda_2 = \frac{K_1^2 K_3^2}{S T}. \quad (\text{D.10})$$

When checking the soft divergences of the seven-point amplitude we need the $1/\epsilon$ singularities arising from soft singularities in the loop integration. For the boxes relevant to the seven-point amplitude these are,

$$\begin{aligned}
I^{abc\{defg\}}|_{1/\epsilon} &= -\frac{2}{s_{ab}s_{bc}(4\pi)^2} \left[\frac{\ln(-s_{ab}) + \ln(-s_{bc}) - \ln(-t_{abc})}{\epsilon} \right], \\
I^{a(bc)(def)g}|_{1/\epsilon} &= -\frac{2}{s_{ag}t_{abc}(4\pi)^2} \left[\frac{\ln(-s_{ag}) + 2\ln(-t_{abc}) - \ln(-s_{bc}) - \ln(-t_{def})}{2\epsilon} \right], \\
I^{a(bc)d(efg)}|_{1/\epsilon} &= -\frac{2}{(t_{abc}t_{bcd} - s_{bc}t_{efg})(4\pi)^2} \left[\frac{\ln(-t_{abc}) + \ln(-t_{bcd}) - \ln(-s_{bc}) - \ln(-t_{efg})}{\epsilon} \right], \\
I^{a(bc)(de)(fg)}|_{1/\epsilon} &= -\frac{2}{(t_{abc}t_{fga} - s_{bc}s_{fg})(4\pi)^2} \left[\frac{\ln(-t_{abc}) + \ln(-t_{fga}) - \ln(-s_{bc}) - \ln(-s_{fg})}{2\epsilon} \right].
\end{aligned} \tag{D.11}$$

D.2 Triangle and Bubble integral Functions

Triangle integral functions may have one, two or three massless legs:



The one-mass triangle depends only on the momentum invariant of the massive leg,

$$I_3^{1m} = \frac{r_\Gamma}{\epsilon^2} (-K_1^2)^{-1-\epsilon}. \tag{D.12}$$

The next integral function is the two-mass triangle integral,

$$I_3^{2m} = \frac{r_\Gamma}{\epsilon^2} \frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)}. \tag{D.13}$$

Note that the one and two mass triangles are linear combinations of the set of functions,

$$G(-K^2) = r_\Gamma \frac{(-K^2)^{-\epsilon}}{\epsilon^2}, \tag{D.14}$$

with,

$$I_3^{1m} = G(-K_1^2), \quad I_3^{2m} = \frac{1}{(-K_1^2) - (-K_2^2)} (G(-K_1^2) - G(-K_2^2)). \tag{D.15}$$

The $G(-K^2)$ are labelled by the independent momentum invariants K^2 and in fact form an independent basis of functions, unlike the one and two-mass triangles which

are not all independent. For example, for six-point kinematics there are only twenty-five independent options for K^2 corresponding to 15 independent s_{ij} 's and 10 independent t_{ijk} 's, whereas there are 15 one-mass triangles and 60 two-mass triangles.

The final scalar triangle is the three-mass integral function. The evaluation of this integral is more involved, and can be obtained from [63, 61],

$$I_3^{3m} = \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^3 \left[\text{Li}_2 \left(- \left(\frac{1+i\delta_j}{1-i\delta_j} \right) \right) - \text{Li}_2 \left(- \left(\frac{1-i\delta_j}{1+i\delta_j} \right) \right) \right] + \mathcal{O}(\epsilon), \quad (\text{D.16})$$

where,

$$\begin{aligned} \delta_1 &= \frac{K_1^2 - K_2^2 - K_3^2}{\sqrt{\Delta_3}}, \\ \delta_2 &= \frac{-K_1^2 + K_2^2 - K_3^2}{\sqrt{\Delta_3}}, \\ \delta_3 &= \frac{-K_1^2 - K_2^2 + K_3^2}{\sqrt{\Delta_3}}, \end{aligned} \quad (\text{D.17})$$

and

$$\Delta_3 \equiv -(K_1^2)^2 - (K_2^2)^2 - (K_3^2)^2 + 2(K_1^2 K_2^2 + K_3^2 K_1^2 + K_2^2 K_3^2). \quad (\text{D.18})$$

Finally, the bubble integral is,

$$I_2(K^2) = \frac{r_\Gamma}{\epsilon(1-2\epsilon)} (-K^2)^{-\epsilon}. \quad (\text{D.19})$$

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